INFATION - 2.

GENERATION OF COSMOLOGICAL PERTURBATIONS.

DENSITY PERTURBATIONS OBSERVED TODAY — FROM VACUUM FLUCTUATIONS OF INFLATON FIELD

GRAVITATIONAL WAVES HOPEFULLY, TO BE OBSERVED — FROM VACUUM FLUCTUATIONS OF GRAVITATIONAL FIELD

⟵ DIGRESSION

INFLATON WITH ITS PERTURBATIONS

$$\varphi(\vec{x},t) = \varphi_c(t) + \phi(\vec{x},t)$$

$\varphi_c(t)$: CLASSICAL SOLUTION, SLOWLY VARIES AT INFLATION

$\phi(\vec{x},t)$: MASSLES SCALAR FIELD IN INFLATING UNIVERSE

NB. SCALAR POTENTIAL NEGLIGIBLE FOR ϕ AT INFLATION

DIGRESSION: QUANTUM MASSLESS SCALAR FIELD IN MINKOWSKI SPACE-TIME.

ACTION

$$S = \int d^4x \left[\frac{1}{2} \eta^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi \right]$$

ENERGY (HAMILTONIAN)

$$H = \int d^3x \left[\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 \right]$$

CONSIDER FIELD IN FINITE BOX OF LENGTH L WITH PERIODIC BOUNDARY CONDITIONS

$$x^1 \in [0, L]$$

$$\varphi(x^1 + L, x^2, x^3) = \varphi(x^1, x^2, x^3)$$

Make FOURIER DECOMPOSITION IN SPACE

$$\varphi(x^1, x^2, x^3) = \sum_{n_1 n_2 n_3} \frac{1}{L^{3/2}} e^{ik_1 x_1 + ik_2 x_2 + ik_3 x_3} \cdot \varphi_{n_1 n_2 n_3}$$

Periodicity: $k_1 = \frac{2\pi}{L} n_1$, $k_2 = \frac{2\pi}{L} n_2$, $k_3 = \frac{2\pi}{L} n_3$

n_i = interger

Factor $\frac{1}{L^{3/2}}$: NORMALIZATION OF WAVE FUNCTION

DIGRESSION CONT'D

$$\varphi(\vec{x}) = \sum_{\vec{n}} \frac{1}{L^{3/2}} e^{i\vec{k}\vec{x}} \varphi_{\vec{k}}$$

$$\vec{k} = \frac{2\pi}{L}\vec{n} \;; \quad \vec{n} = (n_1, n_2, n_3), \text{ integer}$$

Real FIELD: $\varphi(\vec{x}) = \varphi^*(\vec{x}) \Rightarrow \varphi_{\vec{k}} = \varphi^*_{-\vec{k}}$

TECHNICALITY

HAMILTONIAN

$$H = \sum_{\vec{n}} \left(\frac{1}{2} \dot{\varphi}_{\vec{k}} \dot{\varphi}^*_{\vec{k}} + \frac{1}{2} \vec{k}^2 \varphi_{\vec{k}} \cdot \varphi^*_{\vec{k}} \right)$$

SUM OF HAMILTONIANS OF HARMONIC OSCILLATORS, EACH OF FREQUENCY $\omega_k = |\vec{k}|$.

QUANTIZE. HEISENBERG PICTURE.

AT $t=0$

$$\hat{\varphi}_{\vec{k}} = \frac{1}{\sqrt{2\omega_k}} \left(a_{\vec{k}} + a^+_{-\vec{k}} \right)$$

technicality

$$\hat{\pi}_{\vec{k}} \equiv \dot{\hat{\varphi}}^+_k = -i\sqrt{\frac{\omega_k}{2}} \left(a_{-\vec{k}} - a^+_{\vec{k}} \right)$$

WITH

$$[a_{\vec{k}}, a^+_{\vec{k}}] = 1$$

$$[a_{\vec{k}}, a^+_{\vec{k}'}] = 0 \quad \text{FOR } \vec{k} \neq \vec{k}'$$

CREATION AND ANNIHILATION OPERATORS FOR OSCILLATOR EXCITATIONS $\equiv$ SCALAR PARTICLES.

HEISENBERG PICTURE: AT TIME t

$$\hat{\varphi}_{\vec{k}} = \frac{1}{\sqrt{2\omega_k}} \left(a_{\vec{k}} e^{-i\omega_k t} + a^{+}_{-\vec{k}} e^{i\omega_k t} \right)$$

$\omega_k = |\vec{k}|$

$$[a_{\vec{k}}, a^{+}_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \equiv \delta_{n_1,n_1'} \delta_{n_2 n_2'} \delta_{n_3 n_3'}$$

VACUUM = LOWEST ENERGY STATE

= GROUND STATE OF ALL OSCILLATORS

YET THERE EXIST VACUUM FLUCTUATIONS

E.G., EQUAL-TIME CORRELATION FUNCTION

$$\langle 0 | \hat{\varphi}(\vec{x}) \hat{\varphi}(\vec{y}) | 0 \rangle = \frac{1}{L^3} \sum_{\vec{n},\vec{n}'} \langle 0 | \hat{\varphi}_{\vec{k}} \hat{\varphi}_{\vec{k}'} | 0 \rangle \cdot e^{i\vec{k}\vec{x} + i\vec{k}' y}$$

$$\langle 0 | \hat{\varphi}_{\vec{k}} \hat{\varphi}_{\vec{k}'} | 0 \rangle = \frac{1}{2\omega_k} \delta_{\vec{k},-\vec{k}'}$$

$$\Downarrow$$

$$\langle 0 | \hat{\varphi}(\vec{x}) \hat{\varphi}(\vec{y}) | 0 \rangle = \frac{1}{L^3} \sum_{\vec{n}} \frac{1}{2\omega_k} e^{i\vec{k}(\vec{x}-\vec{y})}$$

$\vec{k} = \frac{2\pi}{L} \vec{n}$

INFINITE VOLUME LIMIT

$$\sum_{\vec{n}} = \int dn_1 dn_2 dn_3 = \int \left(\frac{L}{2\pi} dk_1\right) \cdot \left(\frac{L}{2\pi} dk_2\right) \left(\frac{L}{2\pi} dk_3\right)$$

$$= \frac{L^3}{(2\pi)^3} \int d^3 k$$

$$\langle 0|\hat{\varphi}(\vec{x})\hat{\varphi}(\vec{y})|0\rangle = \int \frac{d^3k}{(2\pi)^3\cdot 2\omega_k} e^{i\vec{k}(\vec{x}-\vec{y})}$$

$$\hat{\varphi}(\vec{x},t) = \frac{1}{L^{3/2}} \sum_{\vec{n}} \frac{1}{\sqrt{2\omega_k}} \left(a_{\vec{k}} e^{i\vec{k}\vec{x}-i\omega_k t} + a^+_{\vec{k}} e^{-i\vec{k}\vec{x}+i\omega t}\right)$$

$\Downarrow$ INFINITE VOLUME LIMIT

$$\boxed{\begin{gathered}\hat{\varphi}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_{\vec{k}} e^{i\vec{k}\vec{x}-i\omega_k t} + a^+_{\vec{k}} e^{-i\vec{k}\vec{x}+i\omega_k t}\right) \\ [a_{\vec{k}}, a^+_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}')\end{gathered}}$$

- AMPLITUDES OF VACUUM FLUCTUATIONS

$$\langle 0|[\hat{\varphi}(\vec{x})]^2|0\rangle = \int \frac{d^3k}{(2\pi)^3\cdot 2|\vec{k}|}$$

$$= \frac{4\pi}{(2\pi)^3} \frac{1}{2} \int_{\text{"}0\text{"}}^{\text{"}\infty\text{"}} k^2 \frac{dk}{k}$$

$\Downarrow$

VACUUM FLUCTUATION AT MOMENTUM SCALE k

$$\delta\varphi_{(k)} \simeq \frac{1}{2\pi} k \simeq \frac{1}{\lambda}$$

$\leftarrow$ WAVELENGTH

SMALL AT LONG WAVELENGTH.

PROBLEM FOR TODAY

CONSIDER SCALAR FIELD OPERATOR AT A GIVEN MOMENT OF TIME

$$\hat{\varphi}(\vec{x}) = \int d^3k \left(f_{\vec{k}}(\vec{x})\, a_{\vec{k}} + f^*_{\vec{k}}(\vec{x})\, a^+_{\vec{k}} \right)$$

THIS IS A GAUSSIAN FIELD:

ALL CORRELATION FUNCTIONS ARE EXPRESSED THROUGH TWO-point CORRELATION FUNCTION, PROVIDED THAT

$$[a_{\vec{k}}, a^+_{\vec{q}}] = \delta(\vec{k}-\vec{q}); \quad a_{\vec{k}}|0\rangle = 0$$

ALL OTHER commutators VANISH.

- Find $\langle 0|\hat{\varphi}(\vec{x})\,\hat{\varphi}(\vec{y})|0\rangle$

IN TERMS OF AN INTEGRAL involving $f_{\vec{k}}$

- SHOW THAT

$$\langle 0|\hat{\varphi}(\vec{x})\,\hat{\varphi}(\vec{y})\,\hat{\varphi}(\vec{z})|0\rangle = 0$$

$$\langle 0|\hat{\varphi}(\vec{x})\,\hat{\varphi}(\vec{y})\,\hat{\varphi}(\vec{z})\,\hat{\varphi}(w)|0\rangle =$$

$$\langle 0|\hat{\varphi}(\vec{x})\,\hat{\varphi}(\vec{y})|0\rangle \cdot \langle 0|\hat{\varphi}(\vec{z})\,\hat{\varphi}(\vec{w})|0\rangle + \text{permutations of COORDINATES.}$$

GRAVITATIONAL QUANTUM PERTURBATIONS

GRAVITATIONAL WAVES ARE DESCRIBED BY TRANSVERSE - TRACELESS 3-TENSORS IN MINKOWSKI BACKGROUND

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad , \quad h_{\mu\nu} \ll 1$$

GRAVITATIONAL WAVES HAVE FOURIER COMPONENTS

$$h_{00} = 0 , \qquad h_{0i} = 0$$

$$h_{ij} = h_{\times} e_{ij}^{(\times)} + h_{+} e_{ij}^{(+)}$$

TRANSVERSE TRACELESS POLARIZATION TENSORS $e_{ij}^{(\times,+)}(\vec{k})$

PLUGGING THIS INTO EINSTEIN - HILBERT ACTION, ONE OBTAINS AT QUADRATIC LEVEL

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g}\; R$$

$$= \frac{1}{32\pi G} \sum_A \int d^4x \left[\frac{1}{2} \partial_\mu h_A \partial_\nu h_A \cdot \eta^{\mu\nu} + \mathcal{O}(h^3) \right]$$

$$A = \times , +$$

EACH OF $h_\times , h_+$ BEHAVES AS MASSLESS SCALAR FIELD, EXCEPT FOR NORMALIZATION

$$\tilde{h}_A = \frac{1}{\sqrt{32\pi G}} h_A$$ = PROPERLY NORMALIZED "SCALAR FIELDS"

QUANTIZING $\tilde{h}_A$ IS EXACTLY THE SAME AS THE SCALAR FIELD

$$\tilde{h}_{ij} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \sum_A \left(b^A_{\vec{k}} \, e^{i\vec{k}\vec{x} - i\omega t} + h.c. \right) e^A_{ij}$$

$$h_{ij} = \sqrt{32\pi G} \; \tilde{h}_{ij} = \frac{\sqrt{32\pi}}{M_{pl}} \cdot \tilde{h}_{ij}$$

VACUUM FLUCTUATION AT MOMENTUM SCALE k

two polarizations

$$\delta\tilde{h} = 2 \cdot \frac{1}{2\pi} k$$

$$\Downarrow$$

$$\delta h = \sqrt{\frac{32}{\pi}} \; \frac{k}{M_{pl}} \sim \frac{1}{\lambda M_{pl}}$$

- LARGE AT $k \sim M_{pl}$

GRAVITATIONAL FIELD STRONGLY FLUCTUATES AT $l \lesssim l_{pl} = 10^{-33}$ cm

- Tiny AT LONGER DISTANCE SCALES

$$\delta h \sim \frac{l_{pl}}{\lambda}$$

$\sim 10^{-20}$ at $\lambda \sim 10^{-13}$ cm, PROTON SIZE

$\sim 10^{-57}$ at $\lambda \sim 3 \cdot 10^{24}$ cm $= 1$ Mpc

$$ds^2 = dt^2 - a^2(t)\, dx^i dx^i$$

ACTION AND FIELD EQUATION FOR ϕ

$$S_\phi = \int d^3\vec{x}\, dt \;\; a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2(t)} (\vec{\nabla}\phi)^2 \right]$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2(t)} \nabla^2 \phi = 0$$

FOURIER EXPANSION IN SPACE

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[e^{-i\vec{k}\vec{x}} \phi_k(t) + \text{h.c.} \right]$$

$$\Downarrow$$

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \frac{k^2}{a^2(t)} \phi_k = 0$$

Important: x^i ARE <u>comoving</u> COORDinates
PHYSICAL LENGTH INTERVAL $a(t)dx$

$$\Downarrow$$

$\vec{k}$ ARE <u>CONFORMAL</u> momenta,
time-independent

PHYSICAL MOMENTA ARE

$$\vec{p}(t) = \frac{\vec{k}}{a(t)} \qquad [\, p\cdot(a\,dx) = k\,dx \,]$$

GET REDSHIFTED AS THE
UNIVERSE EXPANDS

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \frac{k^2}{a^2}\phi_k = 0$$

Inflation:

LARGE AT EARLY TIMES

constant

$$H = \frac{\dot{a}}{a}$$

REGIMES: (1) $\frac{k^2}{a^2} \gg H^2$, "SUB-HORIZON"

$$\lambda_{phys} = \frac{1}{p(t)} = \frac{a}{k} \ll H^{-1}$$

Space-time curvature small for this regime.

THIS REGIME OCCURS AT EARLY TIMES.

WAVELENGTH SHORTER THAN (de Sitter) horizon size H^{-1}

SECOND (FRICTION) TERM SMALL; DECAYING OSCILLATIONS WITH TIME-DEPENDENT FREQUENCY

$$\phi_k = \frac{1}{\sqrt{2\omega_k(t)}\; a^{3/2}(t)}\, e^{i\int \omega_k dt} \cdot a_k^+$$

CREATION OPERATOR!

$$\omega_k(t) = \frac{k}{a(t)}$$

NB: $\omega_k(t) \gg H(t)$, field oscillates many times in one HUBBLE TIME H^{-1}. Space-time almost flat.
FACTOR $a^{-3/2}(t)$ compensates a^3 in action

Same expansion in creation-annihilation operators as in flat space, except for slow change of ω in time.

PROBLEM

FOR PURE DE SITTER EXPANSION

$$a(t) = \text{const} \cdot e^{Ht} \quad , \quad H = \text{const}$$

INTRODUCE CONFORMAL TIME η, SUCH THAT

$$dt = a\, d\eta$$

THAT IS

$$\eta = \int \frac{dt}{a(t)}$$

- Find $H(\eta)$ and $a(\eta)$.

Introduce THE FIELD $\chi(\vec{x}, \eta)$ SUCH THAT

$$\phi = \frac{1}{a(\eta)} \chi$$

- SHOW THAT FOR $\frac{k}{a} \gg H$ THE ACTION FOR $\chi(\vec{x},\eta)$ coincides with the action of massless SCALAR FIELD, up TO small CORRECTION.

THUS, χ CAN BE QUANTIZED AT EARLY TIMES IN ABSOLUTELY STANDARD WAY

- SHOW THAT ABOVE EXPRESSION

$$\phi_k = \frac{1}{a^{3/2} \sqrt{2\omega_k}} e^{i\int \omega_k dt} \cdot a_k^+$$

is REPRODUCED THIS WAY.

REGIME 2: $\frac{k}{a} \ll H$, "SUPER-HORIZON"

FRICTION TERM LARGE, FIELD FREEZES IN:

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \cancel{\frac{k^2}{a^2}\phi_k} = 0$$

$$\Downarrow$$

$$\phi_k(t) = \text{const}$$

PROBLEM.

Find the second SOLUTION, NEGLECTING k/a. SHOW THAT IT RAPIDLY DECAYS (AFTER CROSSING OUT THE HORIZON).

MATCH THIS SOLUTION AT TIME WHEN $\frac{k}{a} = H$ TO EARLY-TIME SOLUTION

$$\phi_k = \frac{1}{\sqrt{2\omega_k}\; a^{3/2}}\, e^{i\int \omega_k dt}\, a_k^+$$

$\omega_k = \frac{k}{a}$; $a = k/H \Rightarrow \omega_k = H$ AT MATCHING

$$\sqrt{\omega_k}\cdot a^{3/2} = H^{1/2}\cdot\frac{k^{3/2}}{H^{3/2}} = \frac{k^{3/2}}{H}$$

Up to PHASE

$$\boxed{\phi_k = \frac{H}{\sqrt{2}\, k^{3/2}}\, a_k^+}$$

after crossing OUT THE HORIZON

STAYS CONSTANT OUTSIDE THE HORIZON

WHAT IS SPECIAL ABOUT INFLATION?

- Inflation:

$$\frac{k}{a} \text{ decreases IN TIME , H ALMOST CONSTANT}$$

$$\Downarrow$$

$$\frac{k}{a} \gg H \text{ AT EARLY TIMES ; } \frac{k}{a} \ll H \text{ AT LATE TIME}$$

- Radiation dominated Universe

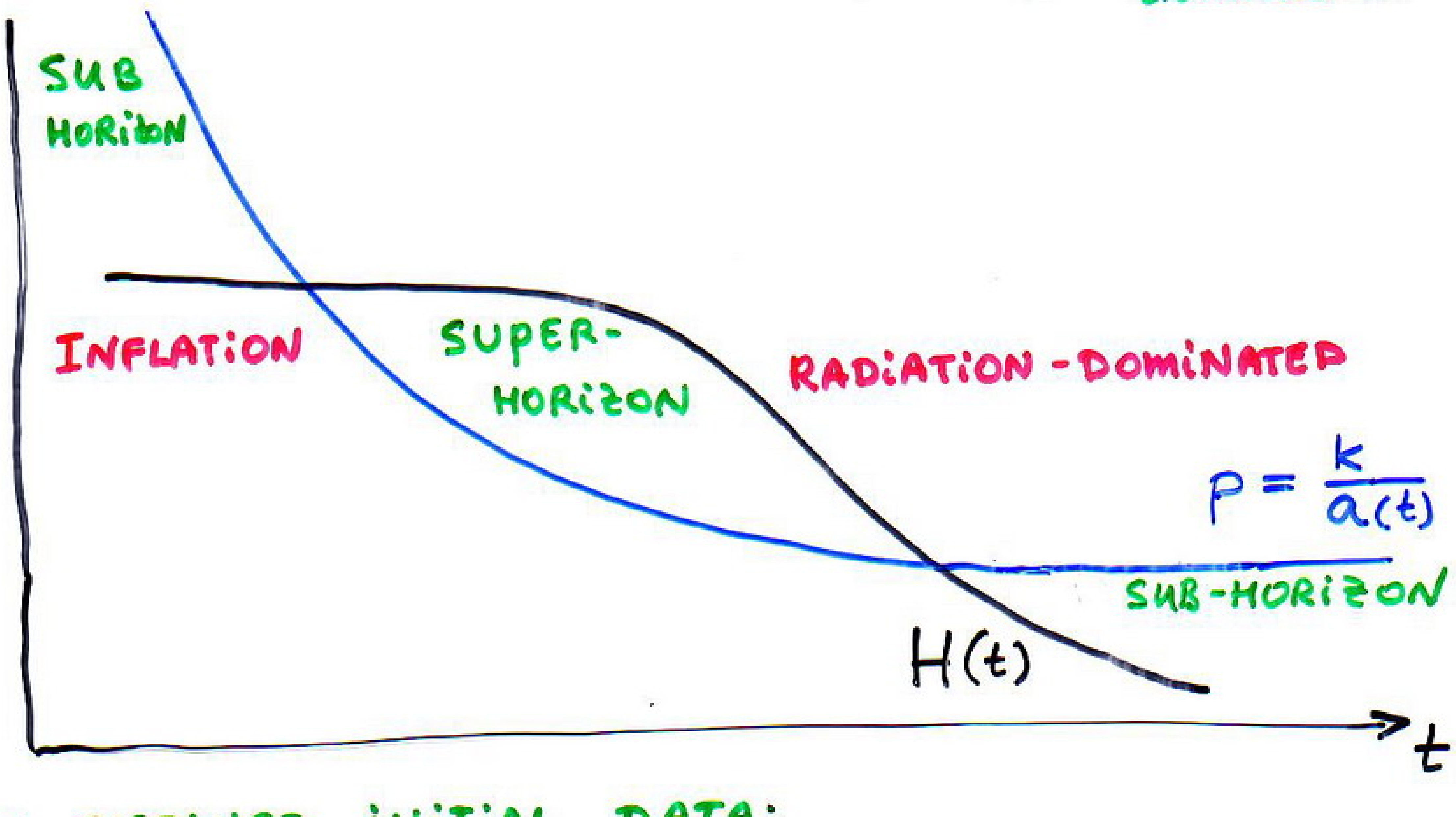
$$a(t) \propto t^{1/2} \quad ; \quad H \propto \frac{1}{t}$$

$$\frac{k}{a} \propto \frac{1}{\sqrt{t}} \quad ; \quad H \propto \frac{1}{t}$$

$$\Downarrow$$

$$\frac{k}{a} \ll H \text{ AT small } t, \quad \frac{k}{a} \gg H \text{ later}$$

Same for matter dominated



WELL-DEFINED INITIAL DATA:
SPACE-TIME effectively flat for a given mode at early times ⇒ reliable calculations.

AFTER CROSSING OUT HORIZON

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \quad \underbrace{\frac{H}{\sqrt{2}\,k^{3/2}}}_{\Uparrow\ \phi_k} \left(a^+_{\vec{k}} + \underbrace{a_{\vec{k}}}_{\Uparrow\ \phi_k^+}\right)$$

STAYS CONSTANT EVEN THOUGH MOMENTUM GETS REDSHIFTED

$$\langle 0|\phi^2(\vec{x})|0\rangle = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2}\,\frac{H^2}{k^3}$$

$$= \frac{H^2}{(2\pi)^2} \int_{\text{"}0\text{"}}^{\text{"}\infty\text{"}} \frac{dk}{k} \equiv \int_0^\infty \Delta_\phi^2(k)\,\frac{dk}{k}$$

FLAT SPECTRUM: EACH DECIMAL PLACE OF MOMENTA GIVES EQUAL CONTRIBUTION

IN FACT, ALMOST FLAT: H SHOULD BE TAKEN AT THE TIME THE MODE k CROSSES OUT HORIZON.

SHORTER WAVELENGTH $\Rightarrow$ LARGER k $\Rightarrow$ LATER $k/a = H$, i.e. LATER HORIZON CROSSING $\Rightarrow$ SMALLER H

$\langle \phi^2 \rangle$ HAS SLIGHTLY RED SPECTRUM.

(MORE AT LONGER WAVELENGTHS THAN AT SHORTER ONES)

ENHANCEMENT AS COMPARED TO MINKOWSKI

$$\delta\phi = \frac{H}{2\pi}$$ DUE TO INFLATION (UNTIL MODE RE-ENTERS HORIZON)

$$\delta\phi_{vac} = \frac{p}{2\pi}$$ IN Minkowski VACUUM

$$\frac{\delta\phi}{\delta\phi_{vac}} = \frac{H}{p}$$

E.G. FOR MODE RE-ENTERING HORIZON TODAY

$$\frac{\delta\phi}{\delta\phi_{vac}} = \frac{H_{inflation}}{H_0}$$

$H_0^{-1} \sim 10^{10}$ yrs $\sim 10^{17}$ s

$H_{infl} \sim \frac{1}{10^{-43}\,s} \cdot \frac{H_{infl}}{M_{Pl}}$

say $\sim 10^{-5}$

$$\Downarrow$$

$$\frac{\delta\phi}{\delta\phi_{vac}} \sim 10^{55} \quad !$$

[OF COURSE, THERE IS NO LONGER INFLATON FIELD PERTURBATIONS TODAY, SEE BELOW]

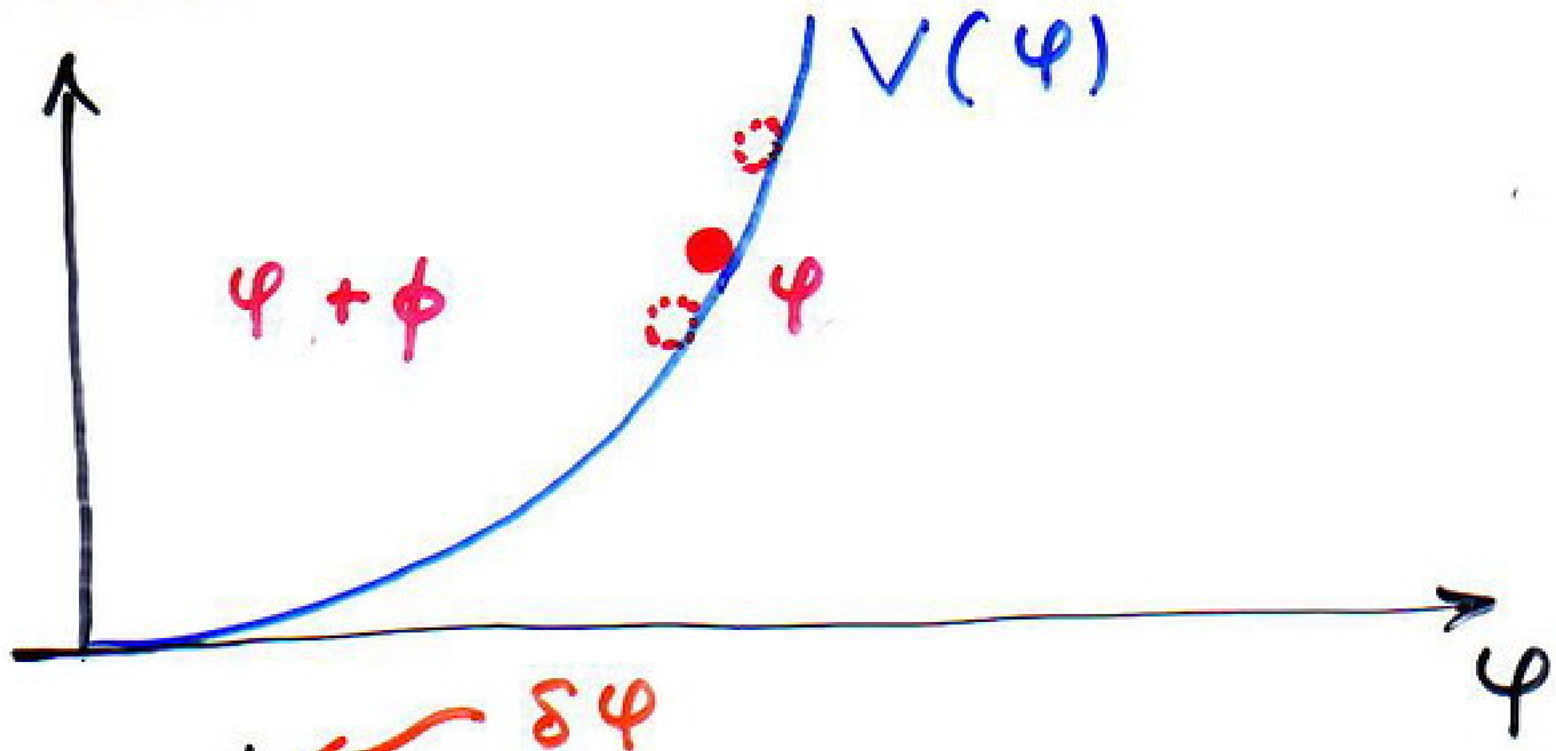
WHY IS THIS RELEVANT FOR DENSITY PERTURBATIONS?

HEURISTIC ARGUMENT GIVING CORRECT RESULT

* PARTS OF THE UNIVERSE OF SIZE H^{-1} EVOLVE INDEPENDENTLY

* IN DIFFERENT PARTS INFLATON FIELD TAKES DIFFERENT VALUES, BECAUSE OF SUPERHORIZON-SIZE PERTURBATIONS OF $\varphi \Rightarrow$

* DIFFERENT PARTS INFLATE FOR DIFFERENT TIME



$$\delta t = -\frac{\phi}{\dot{\varphi}}$$

ϕ ← $\delta\varphi$

$\dot{\varphi}$ ← average field

* EARLIER INFLATION ENDS, MORE DILUTED UNIVERSE

$$\delta\rho = -\dot{\rho}\,\delta t = -H\rho\,\delta t$$

$$\Rightarrow \quad \frac{\delta\rho}{\rho} = \frac{H\,\phi(\vec{x})}{\dot{\varphi}}$$

PRIMORDIAL PERTURBATIONS:

$$\frac{\delta\rho}{\rho} = \text{RANDOM (GAUSSIAN) FIELD}$$

$$\frac{H^2}{(2\pi)^2}\frac{dk}{k}$$

$$< \left[\frac{\delta\rho}{\rho}(\vec{x})\right]^2 > = \int \left(\frac{H}{\dot\varphi}\right)^2 < \phi_k^2 > \frac{d^3k}{(2\pi)^3}$$

$$= \int \left[\frac{H^2}{2\pi\dot\varphi}\right]^2 \frac{dk}{k}$$

def. $\Delta_s^2(k)$

Amplitude of DENSITY PERTURBATIONS

$$\boxed{\Delta_s(k) = \frac{H^2}{2\pi\dot\varphi}}$$

← SLIGHTLY DEPENDS ON k, since taken at time of crossing out the horizon.

AT SLOW ROLL $\quad \dot\varphi = -\frac{V'}{3H} \; ; \; H^2 = \frac{8\pi}{3}\frac{V}{M_{pl}^2}$

$$\Downarrow$$

$$\Delta(k) = -3\frac{H^3}{V'} = 3\left(\frac{8\pi}{3}\right)^{3/2}\frac{V^{3/2}}{V'\cdot M_{pl}^3}$$

CRUDE ESTIMATE FOR AMPLITUDE

$$\Delta_k \sim \frac{10^2}{M_{pl}^3}\left[\frac{V^{3/2}}{V'}\right]_{\varphi\sim M_{pl}}$$

← inflation ends at $\varphi \sim M_{pl}$