# Lecture III: Many-body methods

Morten Hjorth-Jensen[1,2]

[1]Department of Physics and Center of Mathematics for Applications
University of Oslo, N-0316 Oslo

[2]Department of Physics and Astronomy, Michigan State University
East Lansing, Michigan, USA

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# Outline

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# Definitions and notations

We have assumed that the interacting part of the Hamiltonian can be approximated by a two-body interaction. This means that our Hamiltonian is written as

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \sum_{i=1}^{N} h_i + \sum_{i<j=1}^{N} V(r_{ij}), \tag{1}$$

with

$$H_0 = \sum_{i=1}^{N} h_i = \sum_{i=1}^{N} \left( t(\mathbf{r}_i) + u(\mathbf{r}_i) \right). \tag{2}$$

The onebody part $u(\mathbf{r}_i)$ is normally approximated by a harmonic oscillator potential or the Coulomb interaction in case of electronic systems. However, other potentials are fully possible, such as one derived from the self-consistent solution of the Hartree-Fock equations.

Note: I use $N$ for the number of particles.

# Definitions and notations

Our Hamiltonian is invariant under the permutation (interchange) of two particles. Since we deal with fermions however, the total wave function is antisymmetric. Let $\hat{P}$ be an operator which interchanges two particles. Due to the symmetries we have ascribed to our Hamiltonian, this operator commutes with the total Hamiltonian,

$$[\hat{H}, \hat{P}] = 0,$$

meaning that $\Psi_\lambda(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ is an eigenfunction of $\hat{P}$ as well, that is

$$\hat{P}_{ij}\Psi_\lambda(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_N) = \beta\Psi_\lambda(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_N),$$

where $\beta$ is the eigenvalue of $\hat{P}$. We have introduced the suffix $ij$ in order to indicate that we permute particles $i$ and $j$. The Pauli principle tells us that the total wave function for a system of fermions has to be antisymmetric, resulting in the eigenvalue $\beta = -1$.

# Definitions and notations

In our case we assume that we can approximate the exact eigenfunction with a Slater determinant

$$\Phi(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N,\alpha,\beta,\ldots,\sigma) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_\alpha(\mathbf{r}_1) & \psi_\alpha(\mathbf{r}_2) & \ldots & \ldots & \psi_\alpha(\mathbf{r}_N) \\ \psi_\beta(\mathbf{r}_1) & \psi_\beta(\mathbf{r}_2) & \ldots & \ldots & \psi_\beta(\mathbf{r}_N) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \psi_\sigma(\mathbf{r}_1) & \psi_\sigma(\mathbf{r}_2) & \ldots & \ldots & \psi_\gamma(\mathbf{r}_N) \end{vmatrix},$$

where $\mathbf{r}_i$ stand for the coordinates and spin values of a particle $i$ and $\alpha,\beta,\ldots,\gamma$ are quantum numbers needed to describe remaining quantum numbers.

# Definitions and notations

The single-particle function $\psi_\alpha(\mathbf{r}_i)$ are eigenfunctions of the onebody Hamiltonian $h_i$, that is

$$h_i = h(\mathbf{r}_i) = t(\mathbf{r}_i) + u(\mathbf{r}_i),$$

with eigenvalues

$$h_i\psi_\alpha(\mathbf{r}_i) = (t(\mathbf{r}_i) + u(\mathbf{r}_i)\psi_\alpha(\mathbf{r}_i) = \varepsilon_\alpha\psi_\alpha(\mathbf{r}_i).$$

The energies $\varepsilon_\alpha$ are the so-called non-interacting single-particle energies, or unperturbed energies. The total energy is in this case the sum over all single-particle energies, if no two-body or more complicated many-body interactions are present. In many nuclear applications these unperturbed energies are the harmonic oscillator energies.

# Definitions and notations

Let us denote the ground state energy by $E_0$. According to the variational principle we have

$$E_0 \leq E[\Phi] = \int \Phi^* \hat{H} \Phi d\tau$$

where $\Phi$ is a trial function which we assume to be normalized

$$\int \Phi^* \Phi d\tau = 1,$$

where we have used the shorthand $d\tau = d\mathbf{r}_1 d\mathbf{r}_2 \dots d\mathbf{r}_N$.

# Definitions and notations

In the Hartree-Fock method the trial function is the Slater determinant of Eq. (5) which can be rewritten as

$$\Psi(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N,\alpha,\beta,\dots,\nu) = \frac{1}{\sqrt{N!}}\sum_P (-)^P \hat{P}\psi_\alpha(\mathbf{r}_1)\psi_\beta(\mathbf{r}_2)\dots\psi_\nu(\mathbf{r}_N) = \sqrt{N!}\mathcal{A}\Phi_H, \tag{3}$$

where we have introduced the antisymmetrization operator $\mathcal{A}$ defined by the summation over all possible permutations of two nucleons.

# Definitions and notations

It is defined as

$$\mathcal{A} = \frac{1}{N!}\sum_{p}(-)^{p}\hat{P}, \tag{4}$$

with $p$ standing for the number of permutations. We have introduced for later use the so-called Hartree-function, defined by the simple product of all possible single-particle functions

$$\Phi_H(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, \alpha, \beta, \ldots, \nu) = \psi_\alpha(\mathbf{r}_1)\psi_\beta(\mathbf{r}_2)\ldots\psi_\nu(\mathbf{r}_N).$$

# Definitions and notations

Both $\hat{H}_0$ and $\hat{H}_1$ are invariant under all possible permutations of any two particles and hence commute with $\mathcal{A}$

$$[H_0, \mathcal{A}] = [H_1, \mathcal{A}] = 0. \tag{5}$$

Furthermore, $\mathcal{A}$ satisfies

$$\mathcal{A}^2 = \mathcal{A}, \tag{6}$$

since every permutation of the Slater determinant reproduces it.

# Definitions and notations

The expectation value of $\hat{H}_0$

$$\int \Phi^* \hat{H}_0 \Phi d\tau = N! \int \Phi_H^* \mathcal{A} \hat{H}_0 \mathcal{A} \Phi_H d\tau$$

is readily reduced to

$$\int \Phi^* \hat{H}_0 \Phi d\tau = N! \int \Phi_H^* \hat{H}_0 \mathcal{A} \Phi_H d\tau,$$

where we have used eqs. (5) and (6). The next step is to replace the antisymmetrization operator by its definition Eq. (3) and to replace $\hat{H}_0$ with the sum of one-body operators

$$\int \Phi^* \hat{H}_0 \Phi d\tau = \sum_{i=1}^{N} \sum_{p} (-)^p \int \Phi_H^* \hat{h}_i \hat{P} \Phi_H d\tau.$$

# Definitions and notations

The integral vanishes if two or more particles are permuted in only one of the Hartree-functions $\Phi_H$ because the individual single-particle wave functions are orthogonal. We obtain then

$$\int \Phi^* \hat{H}_0 \Phi d\tau = \sum_{i=1}^{N} \int \Phi_H^* \hat{h}_i \Phi_H d\tau.$$

Orthogonality of the single-particle functions allows us to further simplify the integral, and we arrive at the following expression for the expectation values of the sum of one-body Hamiltonians

$$\int \Phi^* \hat{H}_0 \Phi d\tau = \sum_{\mu=1}^{N} \int \psi_\mu^*(\mathbf{r}) \hat{h} \psi_\mu(\mathbf{r}) d\mathbf{r}. \tag{7}$$

# Definitions and notations

We introduce the following shorthand for the above integral

$$\langle \mu | h | \mu \rangle = \int \psi_\mu^*(\mathbf{r}) \hat{h} \psi_\mu(\mathbf{r}),$$

and rewrite Eq. (7) as

$$\int \Phi^* \hat{H}_0 \Phi d\tau = \sum_{\mu=1}^{N} \langle \mu | h | \mu \rangle. \tag{8}$$

# Definitions and notations

The expectation value of the two-body Hamiltonian is obtained in a similar manner. We have

$$\int \Phi^* \hat{H}_1 \Phi d\tau = N! \int \Phi_H^* \mathcal{A} \hat{H}_2 \mathcal{A} \Phi_H d\tau,$$

which reduces to

$$\int \Phi^* \hat{H}_1 \Phi d\tau = \sum_{i \leq j=1}^{N} \sum_p (-)^p \int \Phi_H^* V(r_{ij}) \hat{P} \Phi_H d\tau,$$

by following the same arguments as for the one-body Hamiltonian.

# Definitions and notations

Because of the dependence on the inter-particle distance $r_{ij}$, permutations of any two particles no longer vanish (Slater's rule), and we get

$$\int \Phi^* \hat{H}_1 \Phi d\tau = \sum_{i<j=1}^{N} \int \Phi_H^* V r_{ij})(1 - P_{ij})\Phi_H d\tau.$$

where $P_{ij}$ is the permutation operator that interchanges nucleon $i$ and nucleon $j$.

Again we use the assumption that the single-particle wave functions are orthogonal.

# Definitions and notations

We obtain

$$\int \Phi^* \hat{H}_1 \Phi d\tau = \frac{1}{2} \sum_{\mu=1}^{N} \sum_{\nu=1}^{N} \Big[ \int \psi_\mu^*(\mathbf{r}_i) \psi_\nu^*(\mathbf{r}_j) V(r_{ij}) \psi_\mu(\mathbf{r}_i) \psi_\nu(\mathbf{r}_j) d\mathbf{r}_i \mathbf{r}_j - \int \psi_\mu^*(\mathbf{r}_i) \psi_\nu^*(\mathbf{r}_j) V(r_{ij}) \psi_\nu(\mathbf{r}_i) \psi_\mu(\mathbf{r}_i) d\mathbf{r}_i \mathbf{r}_j \Big] . \tag{9}$$

The first term is the so-called direct term. It is frequently also called the Hartree term, while the second is due to the Pauli principle and is called the exchange term or just the Fock term. The factor 1/2 is introduced because we now run over all pairs twice.

# Definitions and notations

The last equation allows us to introduce some further definitions. The single-particle wave functions $\psi_\mu(\mathbf{r})$, defined by the quantum numbers $\mu$ and $\mathbf{r}$ (recall that $\mathbf{r}$ also includes spin degree) are defined as the overlap

$$\psi_\alpha(\mathbf{r}) = \langle \mathbf{r}|\alpha\rangle.$$

# Definitions and notations

We introduce the following shorthands for the above two integrals

$$\langle \mu\nu | V | \mu\nu \rangle = \int \psi^*_\mu(\mathbf{r}_i)\psi^*_\nu(\mathbf{r}_j)V(r_{ij})\psi_\mu(\mathbf{r}_i)\psi_\nu(\mathbf{r}_j)d\mathbf{r}_i\mathbf{r}_j,$$

and

$$\langle \mu\nu | V | \nu\mu \rangle = \int \psi^*_\mu(\mathbf{r}_i)\psi^*_\nu(\mathbf{r}_j)V(r_{ij})\psi_\nu(\mathbf{r}_i)\psi_\mu(\mathbf{r}_i)d\mathbf{r}_i\mathbf{r}_j.$$

# Definitions and notations

The direct and exchange matrix elements can be brought together if we define the antisymmetrized matrix element

$$\langle \mu\nu|V|\mu\nu\rangle_{AS} = \langle \mu\nu|V|\mu\nu\rangle - \langle \mu\nu|V|\nu\mu\rangle,$$

or for a general matrix element

$$\langle \mu\nu|V|\sigma\tau\rangle_{AS} = \langle \mu\nu|V|\sigma\tau\rangle - \langle \mu\nu|V|\tau\sigma\rangle.$$

It has the symmetry property

$$\langle \mu\nu|V|\sigma\tau\rangle_{AS} = -\langle \mu\nu|V|\tau\sigma\rangle_{AS} = -\langle \nu\mu|V|\sigma\tau\rangle_{AS}.$$

# Definitions and notations

The antisymmetric matrix element is also hermitian, implying

$$\langle \mu\nu|V|\sigma\tau\rangle_{AS} = \langle \sigma\tau|V|\mu\nu\rangle_{AS}.$$

With these notations we rewrite Eq. (9) as

$$\int \Phi^* \hat{H}_1 \Phi d\tau = \frac{1}{2}\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\langle \mu\nu|V|\mu\nu\rangle_{AS}. \tag{10}$$

# Hartree-Fock equations

Use Euler-Lagrange equations and introduce $N^2$ Lagrange multipliers which we denote by $\epsilon_{\mu\nu}$, we can write the variational equation for the energy functional as

$$\delta E - \sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\epsilon_{\mu\nu}\delta\int\psi_\mu^*\psi_\nu = 0.$$

For the orthogonal wave functions $\psi_\mu$ this reduces to

$$\delta E - \sum_{\mu=1}^{N}\epsilon_{\mu}\delta\int\psi_\mu^*\psi_\mu = 0.$$

# Hartree-Fock equations

We can then obtain the standard Hartree-Fock equations

$$\left[-\frac{1}{2}\nabla_i^2 + u(r_i) + \sum_{\nu=1}^{N}\int \psi_\nu^*(\mathbf{r}_j)V(r_{ij})\psi_\nu(\mathbf{r}_j)d\mathbf{r}_j\right]\psi_\mu(\mathbf{x_i})$$
$$-\left[\sum_{\nu=1}^{N}\int \psi_\nu^*(\mathbf{r}_j)V(r_{ij})\psi_\mu(\mathbf{r}_j)d\mathbf{r}_j\right]\psi_\nu(\mathbf{r}_i) = \epsilon_\mu\psi_\mu(\mathbf{r}_i).$$

Not practical for numerical computations.

# Hartree-Fock equations

We prefer to expand the single-particle function in terms of an oscillator basis

$$\psi_a = \sum_{\alpha=0}^{\infty} C_{a\alpha}\phi_\alpha,$$

where $a$ and $\alpha$ represent the relevant single-particle wave functions and $\phi_\alpha$ are the harmonic oscillator functions.

We then vary the coefficients $C_{a\alpha}$. This means that we can precalculate all matrix elements (one-body and two-body) in the basis $\phi_\alpha$.

# Hartree-Fock equations, Exercise

Insert the basis

$$\psi_{h_i} = \sum_{\alpha=0}^{M} C_{h_i\alpha}\phi_\alpha,$$

in the Energy functional (assume a truncation in $M$)

$$E[\Phi] = \sum_h \langle h|h_0(h)|h\rangle + \frac{1}{2}\sum_{h_1}\sum_{h_2}\langle h_1 h_2|V|h_1 h_2\rangle_{AS}.$$

and find the single-particle Hartree-Fock equations.

# Reminder from Lecture II: $V_{\mathrm{low}-k}$

The $A-$body Hamiltonian $H$ is defined as

$$H = \frac{1}{2m}\sum_{i=1}^{A}\mathbf{k}_i^2 + \sum_{i<j}^{A} V_{\mathrm{low-k}}(i,j).$$

Write internal kinetic energy as

$$T_{\mathrm{in}} = T - T_{\mathrm{c.m.}} = \left(1-\frac{1}{A}\right)\sum_{i=1}^{A}\frac{\mathbf{k}_i^2}{2m} - \sum_{i<j}^{A}\frac{\mathbf{k}_i\cdot\mathbf{k}_j}{mA}.$$

The introduction of an additional two-body term yields a modified two-body interaction

$$H_{\mathrm{I}} = V_{\mathrm{low-k}} + V_{\mathrm{c.m.}} = \sum_{i<j}^{A}\left(V_{\mathrm{low-k}}(i,j) - \frac{\mathbf{k}_i\cdot\mathbf{k}_j}{mA}\right).$$

This interaction is in turn written out in terms of harmonic oscillator elements.

# Reminder from Lecture II: $G$-matrix

The $A-$body Hamiltonian $H$ is defined as for the $V\mathrm{low}_k$ case

$$H = \frac{1}{2m}\sum_{i=1}^{A}\mathbf{k}_i^2 + \sum_{i<j}^{A} G(i,j).$$

Internal kinetic energy as

$$T_{\mathrm{in}} = T - T_{\mathrm{c.m.}} = \left(1 - \frac{1}{A}\right)\sum_{i=1}^{A}\frac{\mathbf{k}_i^2}{2m} - \sum_{i<j}^{A}\frac{\mathbf{k}_i\cdot\mathbf{k}_j}{mA}.$$

The modified two-body interaction

$$H_{\mathrm{I}} = G + V_{\mathrm{c.m.}} = \sum_{i<j}^{A}\left(G(i,j) - \frac{\mathbf{k}_i\cdot\mathbf{k}_j}{mA}\right).$$

This interaction is in turn written out in terms of harmonic oscillator elements.

*Both the G-matrix codes and the $V_{\mathrm{low}-k}$ codes list separately $G$ or $V_{\mathrm{low}-k}$ in addition to the term $\frac{\mathbf{k}_i\cdot\mathbf{k}_j}{mA}$. The last term has to be multiplied by $\hbar\omega/A$ in order to be used in derivations of the effective interaction.*

# Reminder from Lecture II: No-core

The total Hamiltonian is

$$H_P^\omega = \sum_{i=1}^{A} P\left[\frac{\vec{p}_i^2}{2m} + \frac{1}{2}m\omega^2\vec{r}_i^2\right]P + \sum_{i<j}^{A} P\left[V_{ij} - \frac{m\omega^2}{2A}(\vec{r}_i - \vec{r}_j)^2\right]_{\rm eff} P$$

$$-PH_{\rm CoM}P.$$

The two-body part of the center-of-mass Hamiltonian is listed separately and needs to be multiplied by $\hbar\omega/A$. Since we only give the two-body part, you need to add the Harmonic oscillator single-particle energies to this part and multiply the harmonic oscillator single-particle energies with $\hbar\omega/A$ as well.

*Note that the no-core Hamiltonian depends explicitely on the mass number A. The G-matrix and $V_{\rm low-}k$ include only a mass dependence via the term $\frac{\mathbf{k}_i\cdot\mathbf{k}_j}{mA}$. The Coulomb interaction can be included in all models.*

# Two-body Matrix Elements I

The renormalized nucleon-nucleon interaction in an arbitrary two-particle basis in the laboratory frame is given by

$$\langle ab|H_{\rm I}|cd\rangle = \langle (n_a l_a j_a t_{z_a})(n_b l_b j_b t_{z_b}) JT_z|\, H_{\rm I}\, |(n_c l_c j_c t_{z_c})(n_d l_d j_d t_{z_d}) JT_z\rangle\,.$$

Here $H_{\rm I}$ can be a $G$-matrix, it can be a no-core or Vlowk interaction. The two-body state $|ab\rangle$ is implicitly coupled to good angular momentum $J$. The labels $n_{a\ldots d}$ number all bound, resonant and discretized scattering states with orbital and angular momenta $(l_{a\ldots d}, j_{a\ldots d})$. Here these single-particle states will be the Hartree-Fock states.
In order to efficiently calculate the matrix elements, we introduce a two-particle harmonic oscillator basis completeness relation

$$\sum_{\alpha\leq\beta} |\alpha\beta\rangle\langle\alpha\beta| = \mathbf{1},$$

where the sum is not restricted in the neutron-proton case. We introduce the greek single particle labels $\alpha, \beta$ for the single-particle harmonic oscillator states in order to distinguish them from the latin single-particle labels Hartree-Fock states $a, b$

# Two-body Matrix Elements II

The interaction can then be expressed in the complete basis is

$$H_{\rm I} = \sum_{\alpha\leq\beta}\sum_{\gamma\leq\delta} |\alpha\beta\rangle\langle\alpha\beta|H_{\rm I}|\gamma\delta\rangle\langle\gamma\delta|,$$

where the sums over two-particle harmonic oscillator states are infinite. The expansion coefficients

$$\langle\alpha\beta|H_{\rm I}|\gamma\delta\rangle = \left\langle (n_\alpha l_\alpha j_\alpha t_{z_\alpha})(n_\beta l_\beta j_\beta t_{z_\beta}) JT_z \right| H_{\rm I} \left| (n_\gamma l_\gamma j_\gamma t_{z_\gamma})(n_\delta l_\delta j_\delta t_{z_\delta}) JT_z \right\rangle ,$$

represent the interaction $H_{\rm I}$ in an antisymmetrized two-particle harmonic oscillator basis, and may easily be calculated using the well known Moshinsky transformation coefficients.

# Two-body Matrix Elements III

The matrix elements are calculated numerically up to $N$ harmonic oscillator two-body states

$$\langle ab|H_{\rm I}|cd\rangle = \sum_{\alpha\leq\beta}^{N}\sum_{\gamma\leq\delta}^{N}\langle ab|\alpha\beta\rangle\langle\alpha\beta|H_{\rm I}|\gamma\delta\rangle\langle\gamma\delta|cd\rangle.$$

The two-particle overlap integrals $\langle ab|\alpha\beta\rangle$ read

$$\langle ab|\alpha\beta\rangle = \frac{\langle a|\alpha\rangle\langle b|\beta\rangle - (-1)^{J-j_\alpha-j_\beta}\langle a|\beta\rangle\langle b|\alpha\rangle}{\sqrt{(1+\delta_{ab})(1+\delta_{\alpha\beta})}}$$

for identical particles (proton-proton or neutron-neutron states) and

$$\langle ab|\alpha\beta\rangle = \langle a|\alpha\rangle\langle b|\beta\rangle$$

for the proton-neutron case.

# Wave Operator

The wave operator $\Omega$ in Rayleigh-Schrödinger perturbation theory can be ordered in terms of the number of interactions with the perturbation $H_1$

$$\Omega = 1 + \Omega^{(1)} + \Omega^{(2)} + \dots,$$

where $\Omega^{(n)}$ means that we have $n$ $H_1$ terms. Explicitly, the above equation reads

$$\begin{aligned}\Omega \left|\psi_\alpha\right\rangle = \quad \left|\psi_\alpha\right\rangle + \sum_i \frac{\left|i\right\rangle \left\langle i\right| H_1 \left|\psi_\alpha\right\rangle}{\varepsilon_\alpha - \varepsilon_i} + \sum_{ij} \frac{\left|i\right\rangle \left\langle i\right| H_1 \left|j\right\rangle \left\langle j\right| H_1 \left|\psi_\alpha\right\rangle}{(\varepsilon_\alpha - \varepsilon_i)(\varepsilon_\alpha - \varepsilon_j)} \\ - \sum_{\beta i} \frac{\left|i\right\rangle \left\langle i\right| H_1 \left|\psi_\beta\right\rangle \left\langle \psi_\beta\right| H_1 \left|\psi_\alpha\right\rangle}{(\varepsilon_\alpha - \varepsilon_i)(\varepsilon_\alpha - \varepsilon_\beta)} + \dots,\end{aligned}$$

where $\varepsilon$ are the unperturbed energies of the $P$-space

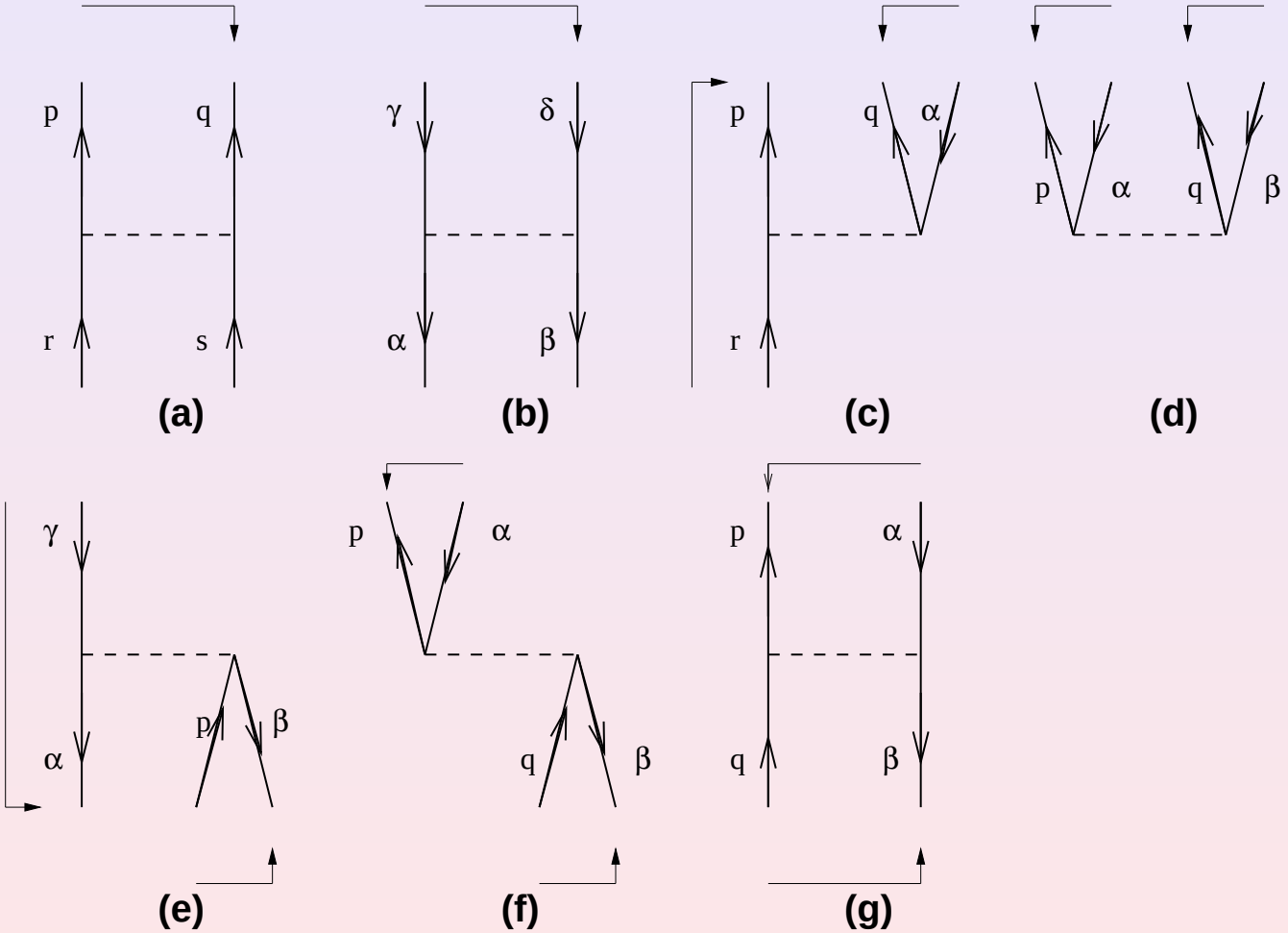
# Wave Operator

The first-order part of the wave operator

$$\sum_i \frac{|i\rangle \langle i| H_1 |\psi_\alpha\rangle}{\varepsilon_\alpha - \varepsilon_i}$$

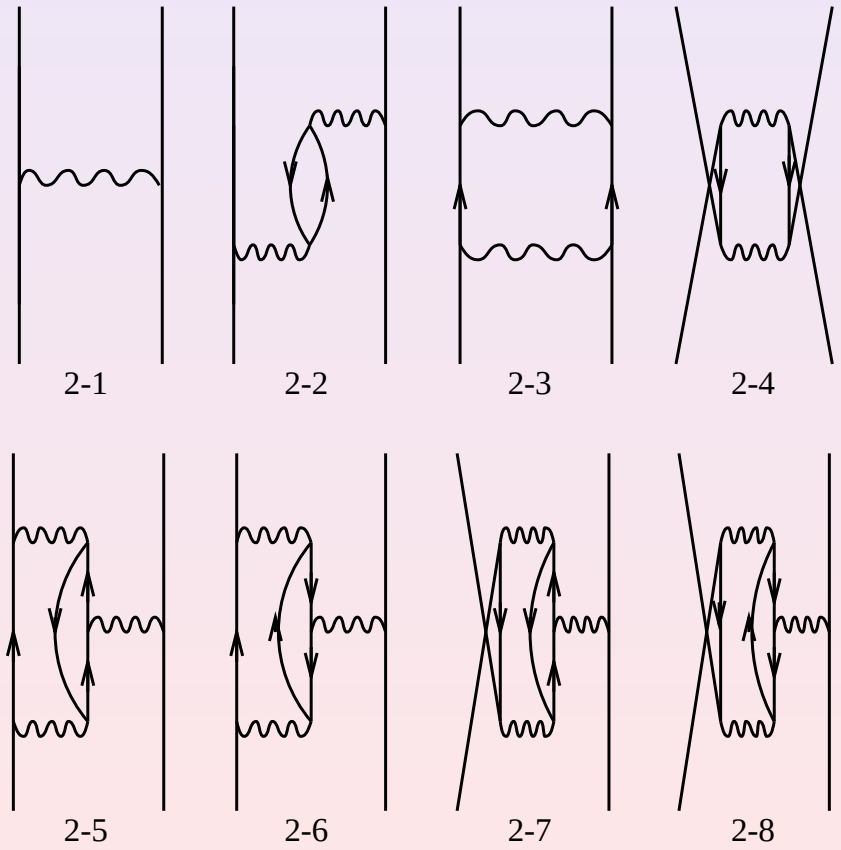
where $\varepsilon$ are the unperturbed energies. How does it look like in a diagrammatic form with a two-body interaction?

# First-order diagrams building up the wave operator



# Perturbation Theory

Order by order perturbation theory in terms of the renormalized interaction.



# Resumming Diagrams

Define the following

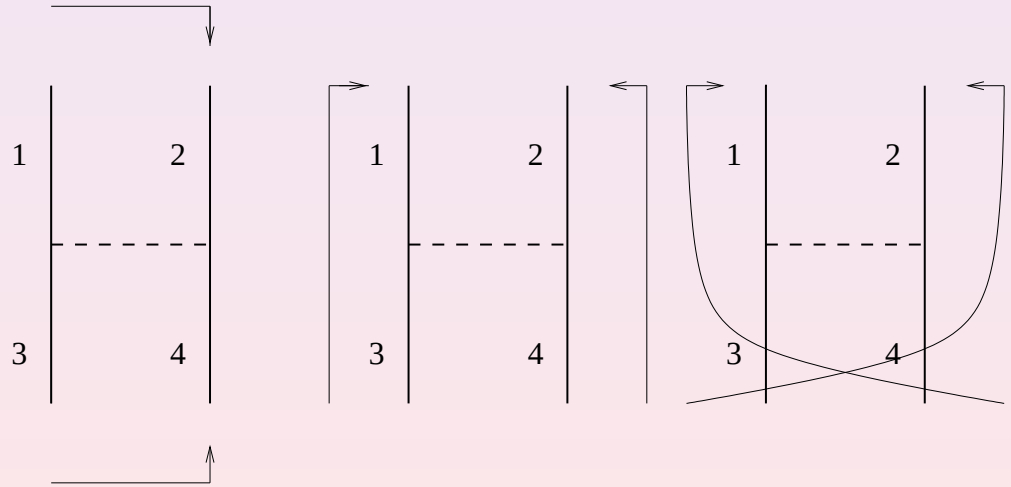
$$V^{[12]}_{1234J} = \langle (12)J | \, V \, | (34)J \rangle$$

$$V^{[13]}_{1234J} = \sum_{J'} (-)^{j_1+j_4+J+J'} \hat{J'}^2 \left\{ \begin{array}{ccc} j_3 & j_1 & J \\ j_2 & j_4 & J' \end{array} \right\} V^{[12]}_{1234J}$$

$$V^{[14]}_{1234J} = \sum_{J'} (-)^{j_1+j_4+J+2j_3} \hat{J'}^2 \left\{ \begin{array}{ccc} j_4 & j_1 & J \\ j_2 & j_3 & J' \end{array} \right\} V^{[12]}_{1234J}$$

$$V^{[14]}_{1234J} = \sum_{J'} (-)^{2j_1+2j_2+2j_3} \hat{J'}^2 \left\{ \begin{array}{ccc} j_4 & j_1 & J \\ j_3 & j_2 & J' \end{array} \right\} V^{[13]}_{1234J}$$

# Coupling Order by Pictures




**(a)** **(b)** **(c)**

# Particle-Particle and Hole-Hole Correlations

Summation of diagrams in the [12] channel, a more general $G$-matrix

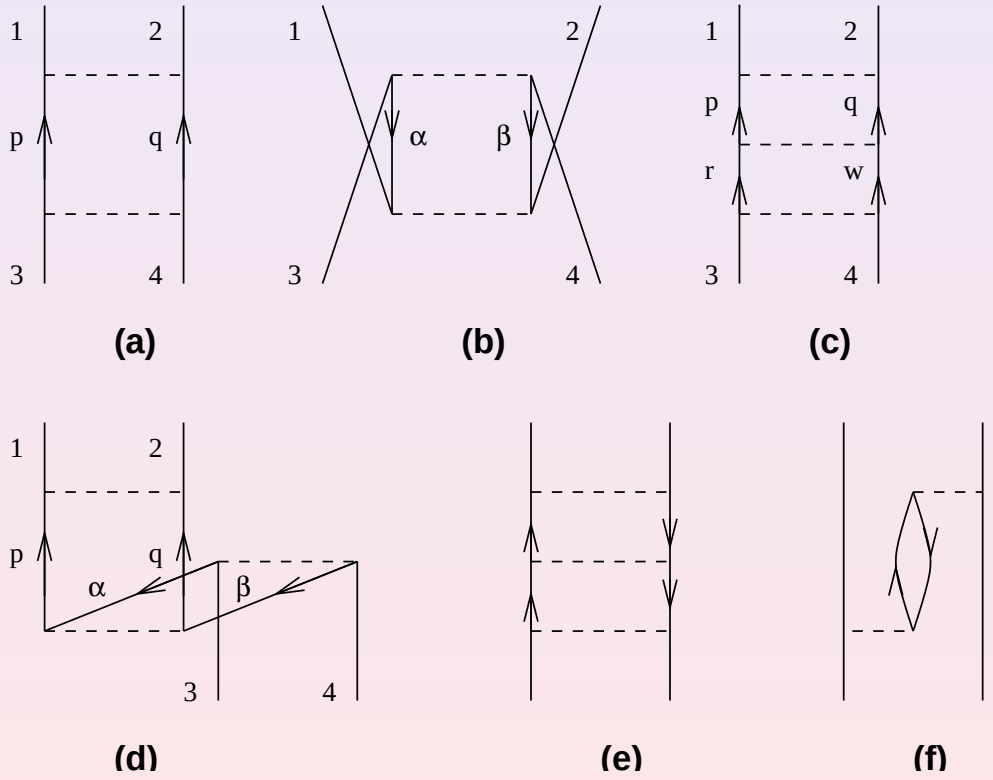
$$\Gamma^{[12]}_{1234J} = \Gamma^{[12]}_{2143J} = -\Gamma^{[12]}_{2134J} = \Gamma^{[12]}_{1243J}$$

$$s = \varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4$$

$$\Gamma^{[12]} = V^{[12]} + V^{[12]}(gg)\Gamma^{[12]}$$

$$\hat{\mathcal{G}}^{[12]} = \frac{Q^{[12]}_{\rm pp}}{s - \varepsilon_5 - \varepsilon_6 + \imath\eta} - \frac{Q^{[12]}_{\rm hh}}{s - \varepsilon_5 - \varepsilon_6 - \imath\eta}$$

# Examples



# Examples: Analytic Expression

$$(a) = \frac{1}{2}\sum_{pq} V^{[12]}_{12pqJ}\frac{1}{s-\varepsilon_p-\varepsilon_q}V^{[12]}_{pq34J}$$

$$(b) = \frac{1}{2}\sum_{\alpha\beta} V^{[12]}_{12\alpha\beta J}\frac{1}{-s+\varepsilon_\alpha+\varepsilon_\beta}V^{[12]}_{\alpha\beta 34J}$$

$$(c) = \frac{1}{4}\sum_{pqrw} V^{[12]}_{12pqJ}\frac{1}{s-\varepsilon_p-\varepsilon_q}V^{[12]}_{pqrwJ}\frac{1}{s-\varepsilon_r-\varepsilon_w}V^{[12]}_{rw34J}$$

# Particle-Hole Diagrams

Screening corrections and vertex renormalization, the equations for the [13] and [14] channels

$$\Gamma^{[13]}_{1234J}(t) = V^{[13]}_{1234J} + \sum_{ph} V^{[13]}_{12phJ} \hat{\mathcal{G}}^{[13]} \Gamma^{[13]}_{ph34J}(t)$$

$$\Gamma^{[14]}_{1234J}(u) = V^{[14]}_{1234J} - \sum_{ph} V^{[14]}_{12phJ} \hat{\mathcal{G}}^{[14]} \Gamma^{[14]}_{ph34J}(u)$$

# Particle-Hole Diagrams

Screening corrections and vertex renormalization, the equations for the [13] and [14] channels

$$\Gamma^{[13]} = V^{[13]} + V^{[13]}(gg)\Gamma^{[13]}$$

$$\Gamma^{[14]} = V^{[14]} + V^{[14]}(gg)\Gamma^{[14]}$$

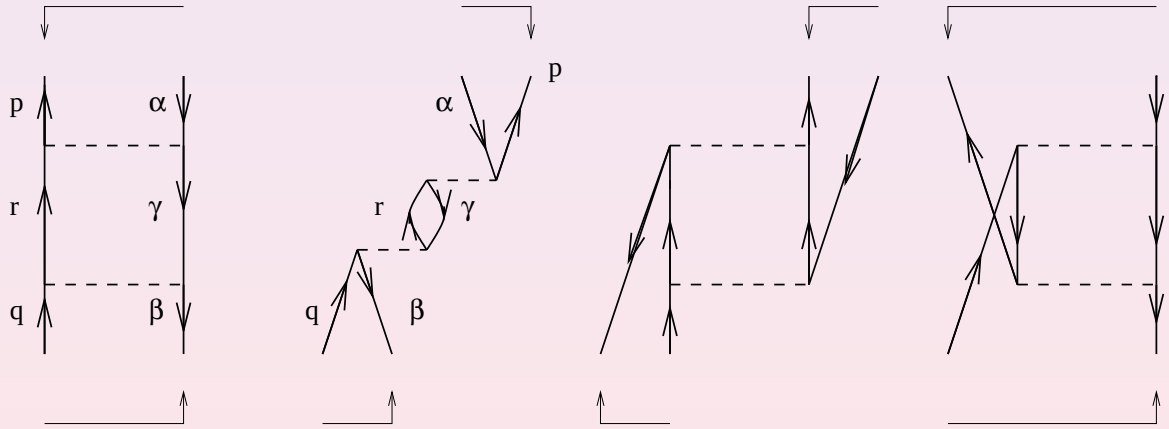
$$t = \varepsilon_3 - \varepsilon_1 = \varepsilon_2 - \varepsilon_4$$

$$u = \varepsilon_1 - \varepsilon_4 = \varepsilon_3 - \varepsilon_2$$

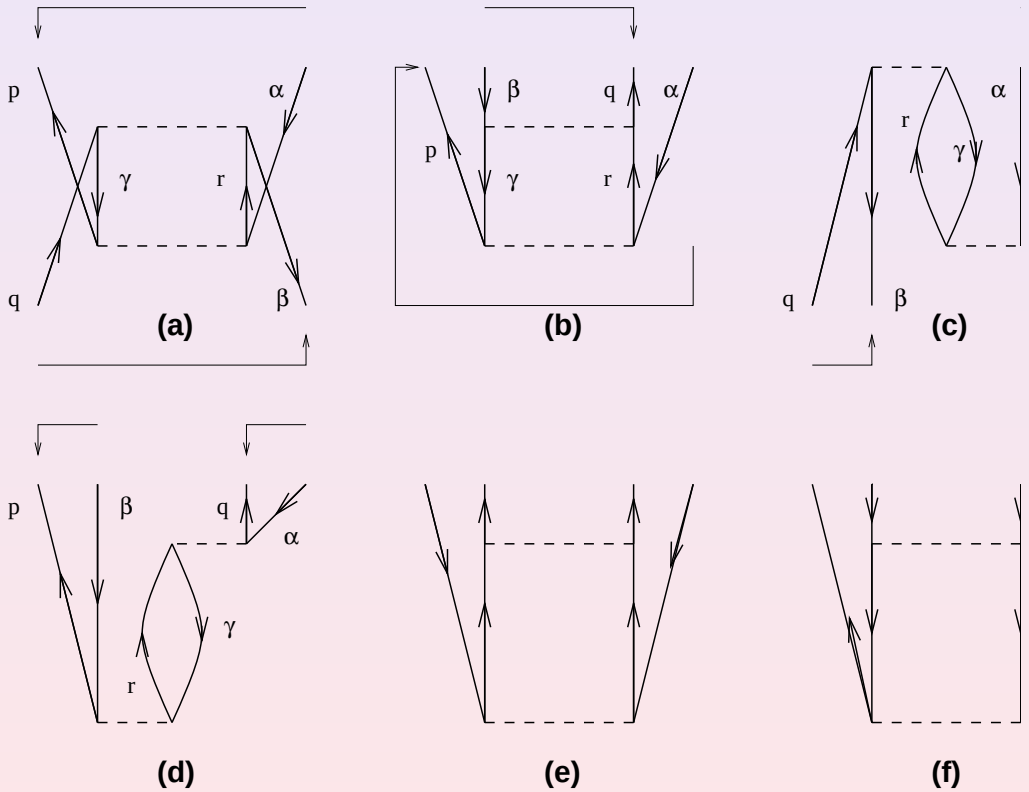
$$\hat{\mathcal{G}}^{[13]} = \frac{Q^{[13]}_{\rm ph}}{t - \varepsilon_p + \varepsilon_h + \imath\eta} - \frac{Q^{[13]}_{\rm hp}}{t + \varepsilon_p - \varepsilon_h - \imath\eta}$$

$$\hat{\mathcal{G}}^{[14]} = \frac{Q^{[14]}_{\rm ph}}{u - \varepsilon_p + \varepsilon_h + \imath\eta} - \frac{Q^{[14]}_{\rm hp}}{u + \varepsilon_p - \varepsilon_h - \imath\eta}$$

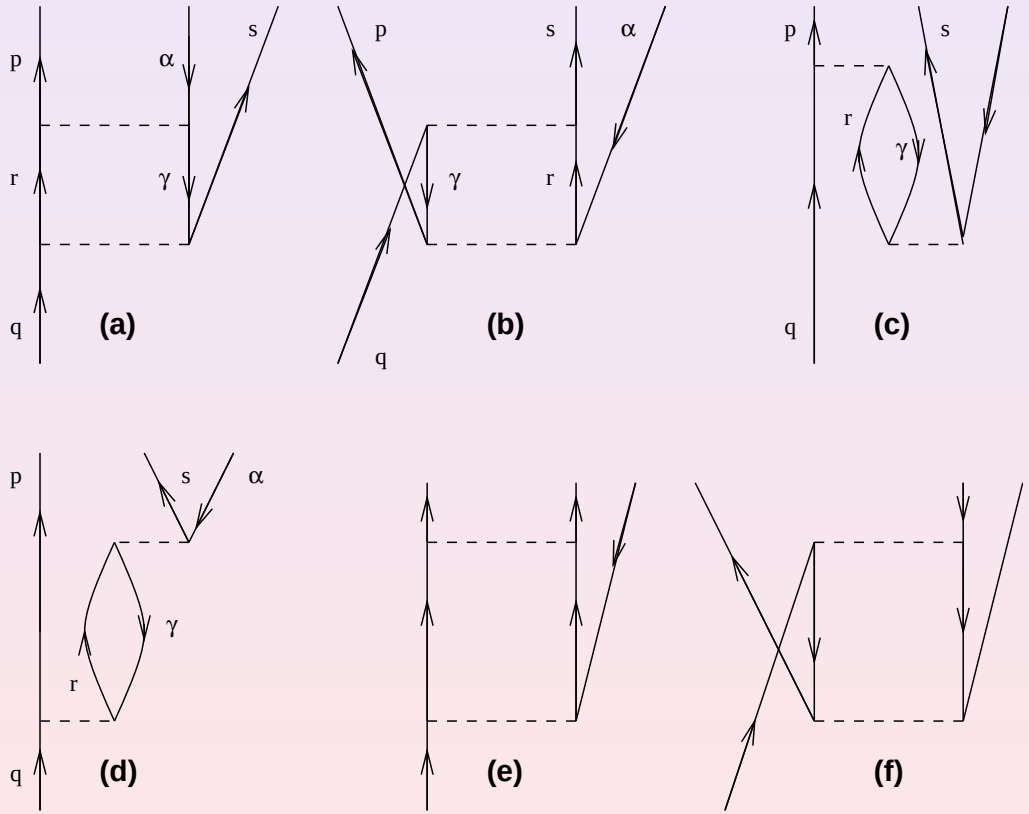
# Particle-Hole Examples



# Particle-Hole Examples



# Particle-Hole Examples



# Effective Operators

In Rayleigh-Schrödinger perturbation theory, the effective interaction $H_{\rm eff}$ can be written out order by order in the interaction $H_1$ as

$$PH_{\rm eff}P = PH_1P + PH_1\frac{Q}{e}H_1P + PH_1\frac{Q}{e}H_1\frac{Q}{e}H_1P + \ldots .$$

Here we have defined $e = \omega - H_0$, where $\omega$ is the so-called starting energy, defined as the unperturbed energy of the interacting particles. Similarly, the exact wave function $|\Psi_\alpha\rangle$ can now be written in terms of the model space wave function as

$$|\Psi_\alpha\rangle = |\Phi_\alpha\rangle + \frac{Q}{e}H_1|\Phi_\alpha\rangle + \frac{Q}{e}H_1\frac{Q}{e}H_1|\Phi_\alpha\rangle + \ldots$$

# Effective Operators

In studies of for example nuclear transitions such as beta decay, the quantity of interest is the transition matrix element between an initial state $|\Psi_i\rangle$ and a final state $|\Psi_f\rangle$ of an operator $\mathcal{O}$ defined as

$$\mathcal{O}_{fi} = \frac{\langle \Psi_f | \mathcal{O} | \Psi_i \rangle}{\sqrt{\langle \Psi_f | \Psi_f \rangle \langle \Psi_i | \Psi_i \rangle}}.$$

Since we perform our calculation in a reduced space, the exact wave functions $|\Psi_{f,i}\rangle$ are not known, only their projections onto the model space. We are then confronted with the problem of how to evaluate $\mathcal{O}_{fi}$ when only the model space wave functions are known. In treating this problem, it is usual to introduce an effective operator $\mathcal{O}^{\text{eff}}_{fi}$, defined by requiring

$$\mathcal{O}_{fi} = \langle \Phi_f | \mathcal{O}_{\text{eff}} | \Phi_i \rangle .$$

# Effective Operators

Observe that $\mathcal{O}_{\rm eff}$ is different from the original operator $\mathcal{O}_{fi}$. The standard empirical procedure is then to introduce some adjustable parameters in $\mathcal{O}_{fi}^{\rm eff}$.
The perturbative expansion for the effective operator can then be written as

$$\langle\Psi_f|\,\mathcal{O}\,|\Psi_i\rangle = \langle\Phi_f|\,\mathcal{O}\,|\Phi_i\rangle + \langle\Phi_f|\,\mathcal{O}\frac{Q}{e}H_1\,|\Phi_i\rangle +$$

$$\langle\Phi_f|\,\frac{Q}{e}H_1\mathcal{O}\,|\Phi_i\rangle + \langle\Phi_f|\,\mathcal{O}\frac{Q}{e}H_1\frac{Q}{e}H_1\,|\Phi_i\rangle + \ldots$$

# Many-Body Perturbation Theory

## Main problems

1. Hard to extend beyond third-order. No systematic way of expanding.
2. No clear signs of convergence in terms of the interaction. Not even in atomic or molecular physics.
3. Difficult to improve upon systematically, e.g., by inclusion of three-body interactions and more complicated correlations.
4. However, enjoys considerable success in producing effective interactions for finite nuclei and the shell model. Good agreement with data.
5. Need non-perturbative resummation techniques for large classes of diagrams. Coupled cluster is one possibility for $A \leq 100$ at present. Can also study Green's function methods (Parquet class of diagrams).

# Reminder of what we typically want to do

Find the lowest ($\approx$ 10–50) solution of the eigenvalue problem for $A$ particles

$$H\left|\Psi_m(A)\right\rangle = (T+V)\left|\Psi_m(A)\right\rangle = E_m\left|\Psi_m(A)\right\rangle$$

and compute other properties with obtained wave functions. Use a valence shell effective Hamiltonian $H_{eff}$ defined within a valence P–space with a pertaining excluded Q–space:

$$P = \sum_{i=1}^{n} \left|\psi_i\right\rangle\left\langle\psi_i\right|, \quad Q = \sum_{i=n+1}^{\infty} \left|\psi_i\right\rangle\left\langle\psi_i\right|.$$

The model space Hamiltonian reads

$$PH_{eff}P\left|\Psi_m\right\rangle = P\left(\widetilde{H}_0 + (H_1)_{eff}\right)P\left|\Psi_m\right\rangle = E_m P\left|\Psi_m\right\rangle$$

# Shell-Model Basics

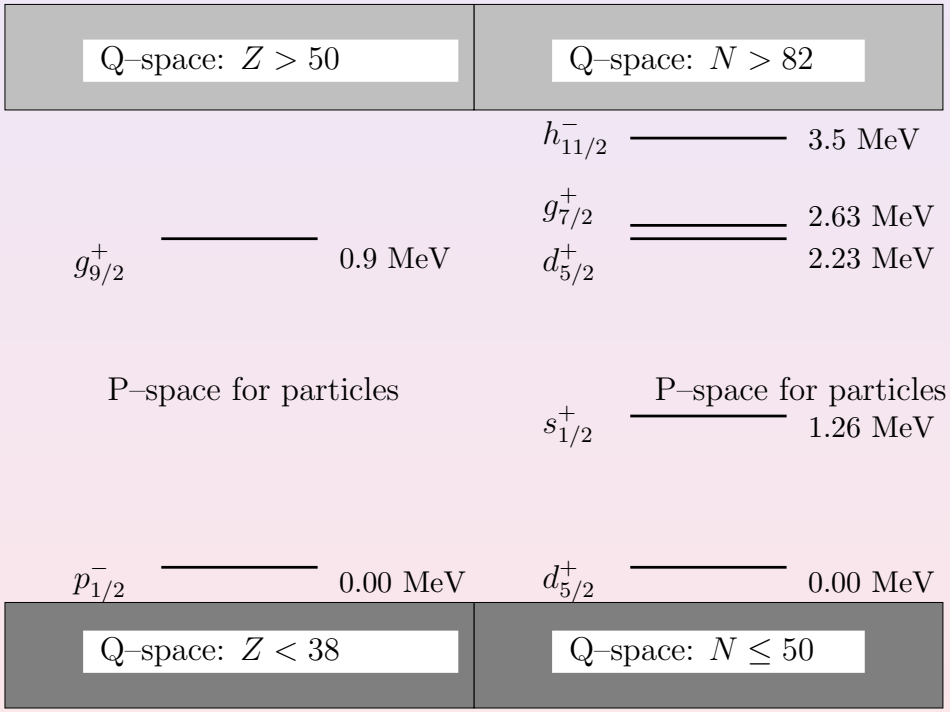
- The choice of basis and the calculation of the matrix elements.

$$\langle \Phi_\lambda | \, P H_{eff} P \, | \Phi_{\lambda'} \rangle = E^0_\lambda \delta_{\lambda,\lambda'} + \langle \Phi_\lambda | \, P (H_1)_{eff} P \, | \Phi_{\lambda'} \rangle$$

  Here we need to have defined the model space and its effective interaction.

- The treatment of giant matrices – diagonalization (Lanczos). Based on the single-particle degrees of freedom which define a model space, we can in turn set up an $A$-body Slater determinant and try to diagonalize.

# Closed Shell Core: $^{88}$Sr, Model Space Example



# Dimensionalities

Solution of the Schrödinger eq. for $N$ nucleons in a valence P–space $\Longrightarrow$ Numerous to infinite degrees of freedom. Number of basic states for the shell model calculation in the Sn isotopes with the single–particle orbits and using the $m$-scheme: $1d_{5/2}$, $0g_{7/2}$, $1d_{3/2}$, $2s_{1/2}$ and $0h_{11/2}$

| System | Dimension | System | Dimension |
|---|---|---|---|
| $^{102}$Sn | 36 | $^{110}$Sn | 1 853 256 |
| $^{103}$Sn | 245 | $^{111}$Sn | 3 608 550 |
| $^{104}$Sn | 1 504 | $^{112}$Sn | 6 210 638 |
| $^{105}$Sn | 7 451 | $^{113}$Sn | 9 397 335 |
| $^{106}$Sn | 31 124 | $^{114}$Sn | 12 655 280 |
| $^{107}$Sn | 108 297 | $^{115}$Sn | 15 064 787 |
| $^{108}$Sn | 323 682 | $^{116}$Sn | 16 010 204 |
| $^{109}$Sn | 828 422 | | |

# More on Dimensionalities

Using $^{100}$Sn as closed shell core as soon as we add protons the dimension grows dramatically

| System | Dimension | System | Dimension |
|---|---|---|---|
| $^{104}$Sn | $\approx 1.5 \cdot 10^3$ | $^{112}$Sn | $\approx 6.2 \cdot 10^6$ |
| $^{108}$Sn | $\approx 3.2 \cdot 10^5$ | $^{116}$Sn | $\approx 1.6 \cdot 10^7$ |
| $^{104}$Sb | $\approx 6.5 \cdot 10^3$ | $^{112}$Sb | $\approx 1.1 \cdot 10^8$ |
| $^{108}$Sb | $\approx 3.2 \cdot 10^6$ | $^{116}$Sb | $\approx 1.9 \cdot 10^9$ |

# Even more on Dimensionalities

Huge dimensionalities in brute force no-core shell-model calcs

| System | 4 major shells | 7 major shells |
|---|---|---|
| $^{4}$He | 4E4 | 9E6 |
| $^{8}$B | 4E8 | 5E13 |
| $^{12}$C | 6E11 | 4E19 |
| $^{16}$O | 3E14 | 9E24 |

Shell-model codes can today reach dimensionalities of $d \sim 10^{10}$ basis states. Monte Carlo based shell-model codes can attack problems with $d \sim 10^{15}$.

# Understanding the Shell Model

We always start with a 'vacuum' reference state, the Slater determinant for the believed dominating configuration of the ground state. Here, Helium as an example; four particles with single-particle wave functions $\phi_i(\mathbf{x}_i)$

$$\Phi_0 = \frac{1}{\sqrt{4!}} \begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_1(\mathbf{x}_2) & \phi_1(\mathbf{x}_3) & \phi_1(\mathbf{x}_4) \\ \phi_2(\mathbf{x}_1) & \phi_2(\mathbf{x}_2) & \phi_2(\mathbf{x}_3) & \phi_2(\mathbf{x}_4) \\ \phi_3(\mathbf{x}_1) & \phi_3(\mathbf{x}_2) & \phi_3(\mathbf{x}_3) & \phi_3(\mathbf{x}_4) \\ \phi_4(\mathbf{x}_1) & \phi_4(\mathbf{x}_2) & \phi_4(\mathbf{x}_3) & \phi_4(\mathbf{x}_4) \end{pmatrix}$$

If this is it, we are staying at the Hartree-Fock level. We can however allow for a linear combination of excitations beyond the ground state, viz., we could assume that we include 1p-1h and 2p-2h excitations

$$\Psi_{2p-2h} = (1 + T_1 + T_2)\Phi_0$$

$T_1$ is a 1p-1h excitation while $T_2$ is a 2p-2h excitation.

Energy diagonalization of giant matrices: Lanczos iteration
Basic operator

$$\begin{aligned} H &= \text{one-particle} + \text{two–particle} = H_0 + V \\ &= \widetilde{V}(N) \\ &= \sum < j_1 m_1 j_2 m_2 | \widetilde{V}(N) | j_3 m_3 j_4 m_4 > a^{\dagger}_{j_1 m_1} a^{\dagger}_{j_2 m_2} a_{j_4 m_4} a_{j_3 m_3} \end{aligned}$$

Example of effective m-scheme two–particle matrix elements outside the $Z = 50 \quad N = 50$ core

| Type | Spherical | m–scheme |
|---|---|---|
| pp or nn | 160 | 5274 |
| pn | 542 | 30105 |

# Shell Model and *m*-Scheme

In a second quantization representation a Slater determinant (SD) is given by

$$|SD_\nu(N)\rangle = \prod_{(jm)\in\nu} a^\dagger_{jm}\,|0\rangle\,,$$

and the complete set is generated by distributing the $N$ particles in all possible ways throughout the basic one–particle states constituting the P–space. This is a very efficient representation. A single $|SD\rangle$ requires only one computer word (32 or 64 bits) and in memory a $|SD\rangle$ with $N$ particles is given by

$$|SD\rangle \longrightarrow (\underbrace{00111101010\cdots}_{N1's}),$$

where each 0 and 1 corresponds to an m–orbit in the valence P–space. Occupied orbits have a 1 and empty orbits a 0.

# Shell Model and *m*-Scheme

Furthermore, all important calculations can be handled in Boolean algebra which is very efficient on modern computers. The action of operators of the form $a^\dagger_\alpha a_\beta$ or $a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta$ acting on an $|SD\rangle$ is easy to perform.
The *m*-scheme allows also for a straightforward definition of many-body operators such as one–, two– and three–particle operators

$$a^\dagger_\alpha a_\beta,$$

$$a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} a_{\beta_1} a_{\beta_2},$$

$$a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} a^\dagger_{\alpha_3} a_{\beta_1} a_{\beta_2} a_{\beta_3},$$

respectively, or generalized seniority operators. The seniority operators can be very useful in preparing a starting vector for the Lanczos iteration process. This option is not included in the program package.

# Shell Model and *m*-Scheme

The generalized seniority operators can then be written as

$$S^\dagger = \sum_j \frac{1}{\sqrt{2j+1}} C_j \sum_{m\geq 0} (-1)^{j-m} a^\dagger_{jm} a^\dagger_{j-m}$$

for seniority zero,

$$D^\dagger_{JM} = \sum_{j\leq j',m,m'} (1+\delta_{j,j'})^{-1/2} \beta_{j,j'} \langle jmj'm' \,|JM\rangle \, a^\dagger_{jm} a^\dagger_{j'm'}$$

for seniority two. The coefficients $C_j$ and $\beta_{jj'}$ can be obtained from the a chosen two-particle system such as the $^{130}$Sn ground state and the excited states, respectively.

# Shell Model and *m*-Scheme

We can also define a seniority four operator

$$\begin{aligned} G(n_1, j_1, n_2, j_2; J, M) &= \left\{ D^\dagger_{n_1,j_1} D^\dagger_{n_2,j_2} \right\}_{J.M=0} \\ &= \sum_{\nu_1 \ldots \nu_4} g^{JM}_{\nu_1 \ldots \nu_4} a^\dagger_{\nu_1} a^\dagger_{\nu_2} a^\dagger_{\nu_3} a^\dagger_{\nu_4} \end{aligned}$$

and a seniority six operator

$$\begin{aligned} I(n_1, j_1, (n_2, j_2, n_3, j_3) j_{23}; J, M) &= \left\{ D_{n_1,j_1} G(n_2, j_2, n_3, j_3; j_{23}) \right\}_{J,M=0} \\ &= \sum_{\nu_1 \ldots \nu_6} g^{JM}_{\nu_1 \ldots \nu_6} a^\dagger_{\nu_1} a^\dagger_{\nu_2} a^\dagger_{\nu_3} a^\dagger_{\nu_4} a^\dagger_{\nu_5} a^\dagger_{\nu_6} \end{aligned}$$

Finally, our shell-model code allows also for the inclusion of effective and real three-body interactions. This version is not included in the program package.

# Lanczos Iteration

Outline of the algorithm:

- We choose an initial Lanczos vector $|lanc_0\rangle$ as the zeroth order approximation to the lowest eigenvector. Our experience is that any reasonable choice is acceptable as long as the vector does not have special properties such as good angular momentum. That would usually terminate the iteration process at too early a stage.
- The next step involves generating a new vector through the process $|new_{p+1} >= H|lanc_p >$. Throughout this process we construct the energy matrix elements of $H$ in this Lanczos basis. First, the diagonal matrix elements of $H$ are then obtained by

$$\langle lanc_p| \, H \, |lanc_p\rangle = \langle lanc_p| \; new_{p+1}\rangle \, ,$$

# Lanczos Iteration

- The new vector $|new_{p+1}\rangle$ is then orthogonalized to all previously calculated Lanczos vectors

$$\left|new'_{p+1}\right\rangle = |new_{p+1}\rangle - |lanc_p\rangle \cdot \langle lanc_p|\ new_{p+1}\rangle - \sum_{q=0}^{p-1} |lanc_q\rangle \cdot \langle lanc_q|\ new_{p+1}\rangle\,,$$

and finally normalized

$$|lanc_{p+1}\rangle = \frac{1}{\sqrt{\left\langle new'_{p+1}\right|\ new'_{p+1}\rangle}} \left|new'_{p+1}\right\rangle\,,$$

to produce a new Lanczos vector.

# Lanczos Iteration

- The off–diagonal matrix elements of $H$ are calculated by

$$\langle lanc_{p+1}| H |lanc_p\rangle = \left\langle new'_{p+1}\right| \left. new'_{p+1}\right\rangle ,$$

and all others are zero.

- After n iterations we have an energy matrix of the form

$$\left\{ \begin{array}{ccccc} H_{0,0} & H_{0,1} & 0 & \cdots & 0 \\ H_{0,1} & H_{1,1} & H_{1,2} & \cdots & 0 \\ 0 & H_{2,1} & H_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & H_{p-1,p} \\ 0 & 0 & 0 & H_{p,p-1} & H_{p,p} \end{array} \right\}$$

as the p'th approximation to the eigenvalue problem.

# Lanczos Iteration

The number p is a reasonably small number and we can diagonalize the matrix by standard methods to obtain eigenvalues and eigenvectors which are linear combinations of the Lanczos vectors.

- This process is repeated until a suitable convergence criterium has been reached.

In this method each Lanczos vector is a linear combination of the basic $|SD\rangle$ with dimension $n$. For $n \approx 10^6 - 10^9$, as in our case of interest. Here is one of the important difficulties associated with the Lanczos method. Large disk storage is needed when the number of Lanczos vector exceeds $\approx 100$. Another difficulty is found in the calculation of $|new_{p+1} >= H|lanc_p >$ when $n > 10^6$.

Problems – Lanczos iteration

- The main cpu time–consuming process

$$H\,|q_i\rangle = |p\rangle$$

due to the large number of non-diagonal matrix elements

$$H\,|q_i\rangle = \sum_{\nu,\mu} C_\nu^i \,\langle SD_\mu|\, H\,|SD_\nu\rangle$$

However, each individual matrix element is easy to calculate

- **Examples:**

| Type | $^{122}$Sn | $^{116}$Sn |
|---|---|---|
| Dimension | $\approx 2 \cdot 10^6$ | $\approx 16 \cdot 10^6$ |
| non-diag. elem | $\approx 4.3 \cdot 10^8$ | $\approx 1.2 \cdot 10^9$ |

Problems – Lanczos iteration

- The matrix elements are needed at every Lanczos iterations Too many non–diagonal two–particle matrix elements to be calculated and saved. Must be recalculated at each iteration
- Numerical roundoff errors requires orthogonalization of all Lanczos vectors. All Lanczos vectors may be kept during the process Slow convergence requires large number of Lanczos vectors ($\geq 100$)
- Due to numerical roundoff errors symmetry properties like angular momentum J are destroyed through the process. However, The final converged eigenvectors have the symmetry properties given by the total Hamiltonian $H$.

# Selected Results

| $^{130}$Sn | | | | $^{128}$Sn | | | |
|---|---|---|---|---|---|---|---|
| $J^\pi$ | Exp. | $J^\pi$ | Theory | $J^\pi$ | Exp. | $J^\pi$ | Theory |
| $(2^+)$ | 1.22 | $2^+$ | 1.46 | $(2^+)$ | 1.17 | $2^+$ | 1.28 |
| $(4^+)$ | 2.00 | $4^+$ | 2.39 | $(4^+)$ | 2.00 | $4^+$ | 2.18 |
| $(6^+)$ | 2.26 | $6^+$ | 2.64 | $(6^+)$ | 2.38 | $6^+$ | 2.53 |
| $^{126}$Sn | | | | $^{124}$Sn | | | |
| $J^\pi$ | Exp. | $J^\pi$ | Theory | $J^\pi$ | Exp. | $J^\pi$ | Theory |
| $2^+$ | 1.14 | $2^+$ | 1.21 | $2^+$ | 1.13 | $2^+$ | 1.17 |
| $4^+$ | 2.05 | $4^+$ | 2.21 | $4^+$ | 2.10 | $4^+$ | 2.26 |
| | | $6^+$ | 2.61 | | | $6^+$ | 2.70 |
| $^{122}$Sn | | | | $^{120}$Sn | | | |
| $J^\pi$ | Exp. | $J^\pi$ | Theory | $J^\pi$ | Exp. | $J^\pi$ | Theory |
| $2^+$ | 1.14 | $2^+$ | 1.15 | $2^+$ | 1.17 | $2^+$ | 1.14 |
| $4^+$ | 2.14 | $4^+$ | 2.30 | $4^+$ | 2.19 | $4^+$ | 2.30 |
| $6^+$ | 2.56 | $6^+$ | 2.78 | | | $6^+$ | 2.86 |
| $^{118}$Sn | | | | $^{116}$Sn | | | |
| $J^\pi$ | Exp. | $J^\pi$ | Theory | $J^\pi$ | Exp. | $J^\pi$ | Theory |
| $2^+$ | 1.22 | $2^+$ | 1.15 | $2^+$ | 1.30 | $2^+$ | 1.17 |

# Seniority Analysis

Seniority $v = 0$ overlap $|\langle {}^{A}Sn; 0^{+}|(S^{\dagger})^{\frac{n}{2}}|\tilde{0}\rangle|^2$ and the seniority $v = 2$ overlaps $|\langle {}^{A}Sn; J_f|D^{\dagger}_{JM}(S^{\dagger})^{\frac{n}{2}-1}|\tilde{0}\rangle|^2$ for the lowest–lying eigenstates of $^{128-120}$Sn.

| | A=128 | A=126 | A=124 | A=122 | A=120 |
|---|---|---|---|---|---|
| $0^{+}_{1}$ | 0.96 | 0.92 | 0.87 | 0.83 | 0.79 |
| $2^{+}_{1}$ | 0.92 | 0.89 | 0.84 | 0.79 | 0.74 |
| $4^{+}_{1}$ | 0.73 | 0.66 | 0.44 | 0.13 | 0.00 |
| $4^{+}_{2}$ | 0.13 | 0.18 | 0.39 | 0.66 | 0.74 |
| $6^{+}_{1}$ | 0.81 | 0.85 | 0.83 | 0.79 | 0.64 |

# Partial Waves and Spectra

$2_1^+ - 0_1^+$ excitation energy for the even tin isotopes $^{130-116}$Sn for various approaches to the effective interaction.

| | $^{116}$Sn | $^{118}$Sn | $^{120}$Sn | $^{122}$Sn | $^{124}$Sn | $^{126}$Sn | $^{128}$Sn | $^{130}$Sn |
|---|---|---|---|---|---|---|---|---|
| Expt | 1.29 | 1.23 | 1.17 | 1.14 | 1.13 | 1.14 | 1.17 | 1.23 |
| $V_{\rm eff}$ | 1.17 | 1.15 | 1.14 | 1.15 | 1.14 | 1.21 | 1.28 | 1.46 |
| $G$-matrix | 1.14 | 1.12 | 1.07 | 0.99 | 0.99 | 0.98 | 0.98 | 0.97 |
| $^1S_0$ $G$-matrix | 1.38 | 1.36 | 1.34 | 1.30 | 1.25 | 1.21 | 1.19 | 1.18 |
| No $^1S_0$ & $^3P_2$ in $G$ | | | | | 0.15 | -0.32 | 0.02 | -0.21 |

# Shell-Model Studiesof Nuclei around $A = 132$

Third-order effective interaction with $G$-matrix for $^{132}$Sn and effective interaction consisting of a model space with

1. $(0g_{7/2}, 1d_{5/2}, 1d_{3/2}, 2s_{1/2}, 0h_{11/2})^{Z-50}$ for proton particles
2. $(0g_{7/2}, 1d_{5/2}, 1d_{3/2}, 2s_{1/2}, 0h_{11/2})^{N-82}$ for neutron holes.
3. The wave functions for $N \geq 82$ were obtained with the same model space for protons as above and with a model space for neutrons of $(0h_{9/2}, 1f_{7/2}, 1f_{5/2}, 2p_{3/2}, 2p_{1/2}, 0i_{13/2})^{N-82}$

# Spectra

## Results: $^{132}$Te

| $J^\pi$ | Experiment | CD-Bonn |
|---|---|---|
| $0_1^+$ | 0.0 | 0.0 |
| $2_1^+$ | 0.97 | 0.95 |
| $(2)_2^+$ | 1.66 | 1.64 |
| $4_1^+$ | 1.67 | 1.54 |
| $6_1^+$ | 1.77 | 1.68 |
| $0_2^+$ | | 1.70 |
| $(2)_3^+$ | 1.79 | 1.93 |
| $(7)_1^-$ | 1.92 | 1.88 |
| $(5)_1^-$ | 2.05 | 2.01 |
| $4_1^-$ | | 2.12 |

## Results: $^{134}$Te

| $J^\pi$ | Experiment | CD-Bonn |
|---|---|---|
| $0_1^+$ | 0.0 | 0.0 |
| $2_1^+$ | 1.28 | 1.21 |
| $4_1^+$ | 1.57 | 1.48 |
| $6_1^+$ | 1.69 | 1.61 |
| $6_2^+$ | 2.40 | 2.17 |
| $2_2^+$ | 2.46 | 2.45 |
| $4_2^+$ | 2.55 | 2.45 |
| $1_1^+$ | 2.63 | 2.41 |
| $3_1^+$ | 2.68 | 2.54 |
| $5_1^+$ | 2.73 | 2.54 |

# Results: Magnetic Moments

| Nuclide | $J^\pi$ | Experiment | Effective | Free | proton | neutron |
|---|---|---|---|---|---|---|
| $^{134}$Sn | $2^+$ | | −0.469 | −0.745 | 0 | −0.469 |
| $^{130}$Sn | $2^+$ | | −0.275 | −0.385 | 0 | −0.275 |
| $^{128}$Sn | $2^+$ | | −0.253 | −0.343 | 0 | −0.253 |
| $^{126}$Sn | $2^+$ | | −0.262 | −0.355 | 0 | −0.262 |
| $^{124}$Sn | $2^+$ | −0.3(2) | −0.270 | −0.364 | 0 | −0.270 |
| $^{136}$Te | $2^+$ | | 0.695 | 0.544 | 0.846 | −0.151 |
| $^{134}$Te | $2^+$ | | 1.724 | 1.035 | 1.724 | 0 |
| $^{132}$Te | $2^+$ | 0.70(10) | 0.975 | 0.575 | 1.027 | −0.052 |
| $^{130}$Te | $2^+$ | 0.59(7) | 0.693 | 0.360 | 0.806 | −0.113 |
| $^{134}$Xe | $2^+$ | 0.708(14) | 0.825 | 0.541 | 0.886 | −0.061 |
| $^{136}$Xe | $2^+$ | 1.53(9) | 1.823 | 1.165 | 1.823 | 0 |
| $^{138}$Xe | $2^+$ | | 0.775 | 0.623 | 0.912 | −0.137 |
| $^{138}$Ba | $2^+$ | 1.44(22) | 2.00 | 1.52 | 2.00 | 0 |