In principle, resonances and decaying particles are different entities. Usually, resonance refers to the energy distribution of the outgoing particles in a scattering process, and it is characterized by its energy and width. A decaying state is described in a time dependent setting by its energy and lifetime. Both concepts are related by:

$$T_0 = \frac{\hbar}{\Gamma}$$

This relation has been checked in numerous precision experiments.

See more discussion in
R. de la Madrid,
Nucl. Phys. A812, 13 (2008)

TABLE III. Recent theoretical and experimental lifetimes τ for NaI $3p\ ^2P_{1/2}$ and $^2P_{3/2}$ and total line strengths $S(3s\text{-}3p)$ (uncertainties given in parentheses).

Ref.	Method	J	τ_J (ns)	S (a.u.)
		Theoretical		
[6]	Semiempirical			37.03
[7]	Semiempirical			37.19
[4]	RMBPT all orders			37.38(11)
[11]	Coupled clusters			37.56[a]
[3]	MCHF-CCP			37.30[a]
[5]	MCHF+CI			37.26[a]
		Experimental		
[1]	BGLS	1/2	16.40(3)	37.04(7)[b]
[16]	Pulsed laser	1/2	16.35(6)	37.15(14)[b]
[17]	C_3 analysis	1/2	16.31(6)	37.24(12)[b]
[18]	Linewidth	3/2	16.237(35)	37.30(8)[b]
This	BGLS	1/2	16.299(21)	37.26(5)[c]
work		3/2	16.254(22)	

[a]Corrected for relativistic effects (−0.09 a.u.) using the ratio between DF and HF values. The original value of Ref. [3] without relativistic correction is 37.39 a.u.
[b]A line strength ratio between the two fine-structure components of 0.5 was assumed in the calculation.
[c]The ratio of the line strengths of the two fine-structure components was determined to 0.50014(44). This is in excellent agreement with the nonrelativistic prediction of 0.5. In the uncertainty estimate for the ratio all those systematical effects were omitted which affect both lifetimes in the same way.

U. Volz et al., Phys. Rev. Lett 76, 2862(1996)

Rigged Hilbert Space: the natural framework to formulate quantum mechanics

In mathematics, a rigged Hilbert space (Gel'fand triple, nested Hilbert space, equipped Hilbert space) is a construction designed to link the distribution and square-integrable aspects of functional analysis. Such spaces were introduced to study spectral theory in the broad sense. They can bring together the 'bound state' (eigenvector) and 'continuous spectrum', in one place.

Mathematical foundations in the 1960s by Gel'fand et al. who combined Hilbert space with the theory of distributions. Hence, the RHS, rather than the Hilbert space alone, is the natural mathematical setting of Quantum Mechanics

I. M. Gel'fand and N. J. Vilenkin. Generalized Functions, vol. 4: Some Applications of Harmonic Analysis. Rigged Hilbert Spaces. Academic Press, New York, 1964.

The resonance amplitude associated with the Gamow states is proportional to the complex delta function and such amplitude can be approximated in the near resonance region by the Breit-Wigner amplitude (Nucl. Phys. A812, 13 (2008)):

$$\mathcal{A}(E_n \to E) \propto i\sqrt{2\pi}\,\delta(E - E_n) \approx -\frac{1}{2\pi}\frac{1}{E - E_n}$$

For a pedagogical description, see R. de la Madrid, Eur. J. Phys. 26, 287 (2005)

Theoretical Approaches

Continuum Shell Model -an old tool!

- U. Fano, Phys. Rev. 124, 1866 (1961)
- C. Mahaux and H. Weidenmüller: "Shell Model Approach to Nuclear Reactions" 1969
- H. W. Bartz et al., Nucl. Phys. A275, 111 (1977)
- D. Halderson and R.J. Philpott, Nucl. Phys. A345, 141
- …
- J. Okolowicz, M. Ploszajczak, I. Rotter, Phys. Rep. 374, 271 (2003)

Recent Developments:

SMEC

- K. Bennaceur et al., Nucl. Phys. A651, 289 (1999)
- K. Bennaceur et al., Nucl. Phys. A671, 203 (2000)
- N. Michel et al., Nucl. Phys. A703, 202 (2002)
- Y. Luo et al., nucl-th/0201073

Gamow Shell Model

- N. Michel et al., Phys. Rev. Lett. 89, 042502 (2002)
- N. Michel et al., Phys. Rev. C67, 054311 (2003)
- N. Michel et al., Phys. Rev. C70, 064311 (2004)
- R. Id Betan et al., Phys. Rev. Lett. 89, 042501 (2002)
- R. Id Betan et al., Phys. Rev. C67, 014322 (2003)
- G. Hagen et al, Phys. Rev. C71, 044314 (2005)

Other approaches

IOP PUBLISHING JOURNAL OF PHYSICS G: NUCLEAR AND PARTICLE PHYSICS

J. Phys. G: Nucl. Part. Phys. 36 (2009) 013101 (40pp) doi:10.1088/0954-3899/36/1/013101

TOPICAL REVIEW

Shell model in the complex energy plane

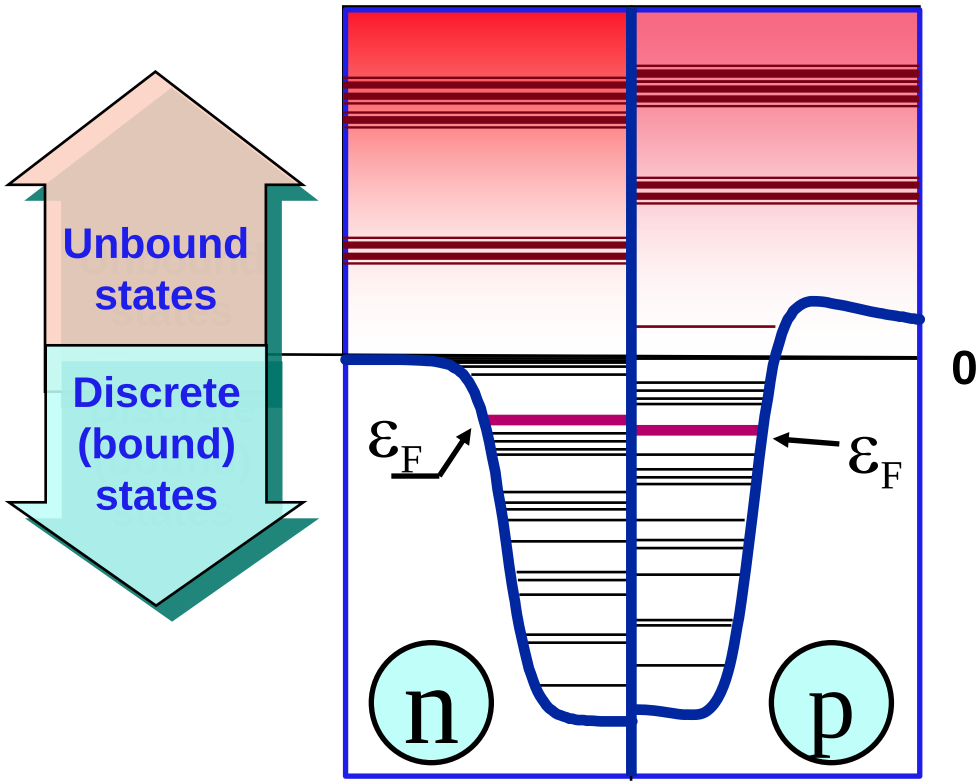
N Michel[1,2], W Nazarewicz[3,4,5], M Płoszajczak[6] and T Vertse[7,8]

Abstract

This work reviews foundations and applications of the complex-energy continuum shell model that provides a consistent many-body description of bound states, resonances and scattering states. The model can be considered a quasi-stationary open quantum system extension of the standard configuration interaction approach for well-bound (closed) systems.

Complex Energy Treatment

Gamow states of a finite potential



Resonant (Gamow) states

Also true in many-channel case!

$$\hat{H}\Psi = \left(e - i\frac{\Gamma}{2}\right)\Psi$$

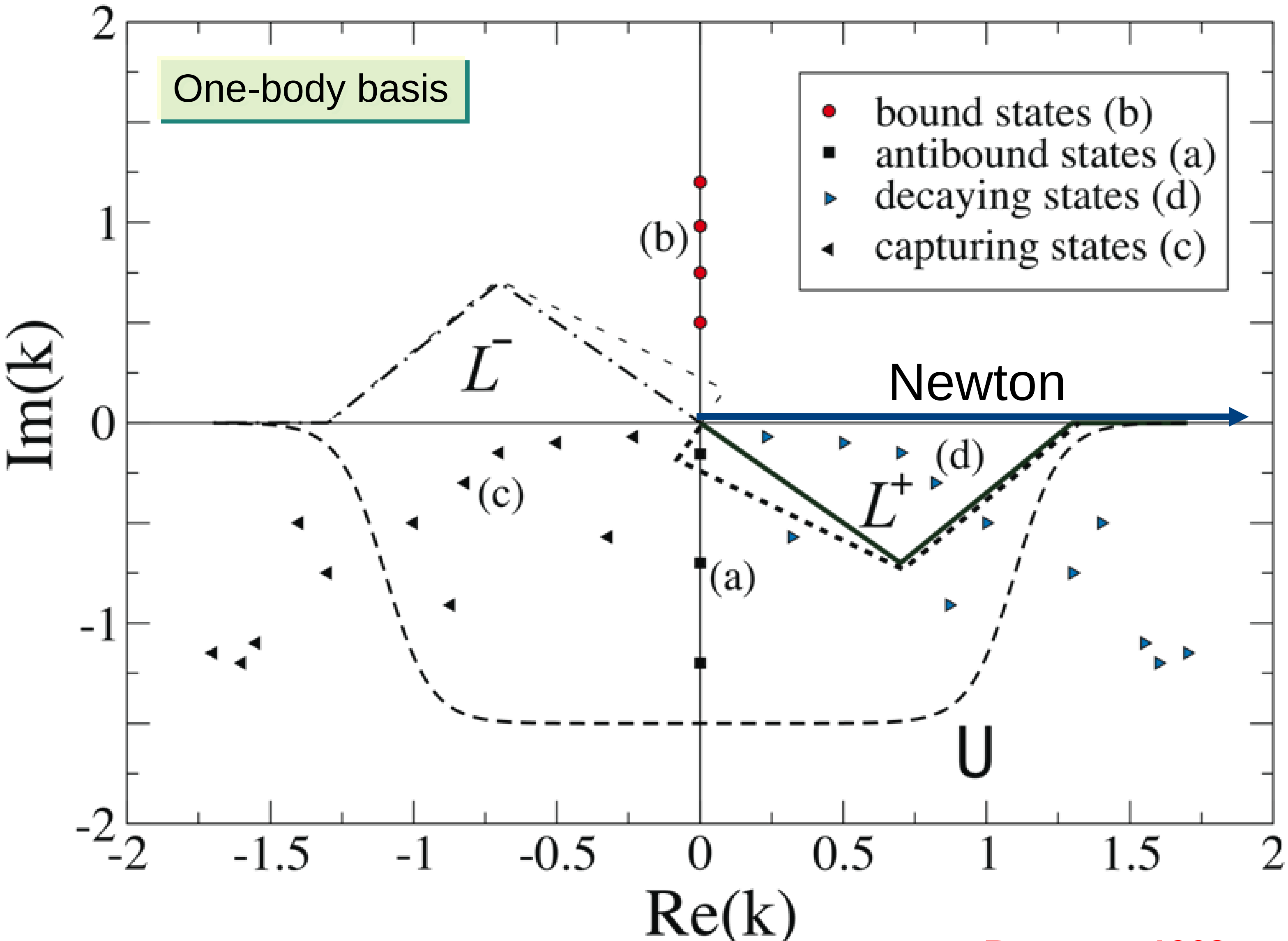
outgoing solution

$$\Psi(0,k) = 0, \quad \Psi(\vec{r},k) \xrightarrow[r\to\infty]{} O_l(kr)$$

$$k_n = \sqrt{\frac{2m}{\hbar^2}\left(e_n - i\frac{\Gamma_n}{2}\right)}$$

complex pole of the S-matrix

- Humblet and Rosenfeld, Nucl. Phys. **26**, 529 (1961)
- Siegert, Phys. Rev. **36**, 750 (1939)
- **Gamow, Z. Phys. 51, 204 (1928)**



Berggren 1968

Berggren ensemble for a given *jl* channel:

$$\sum_{n \in (b,d)} |u_n\rangle\langle u_n| + \int_{L^+} |u(k)\rangle\langle u(k)| dk = 1.$$

Gamow states and completeness relations

T. Berggren, Nucl. Phys. A109, 265 (1968); A389, 261 (1982)
T. Lind, Phys. Rev. C47, 1903 (1993)

$$\sum_{n\in(b,d)} |u_n\rangle\langle u_n| + \int_{L^+} |u(k)\rangle\langle u(k)|dk = 1.$$

Particular case: Newton completeness relation

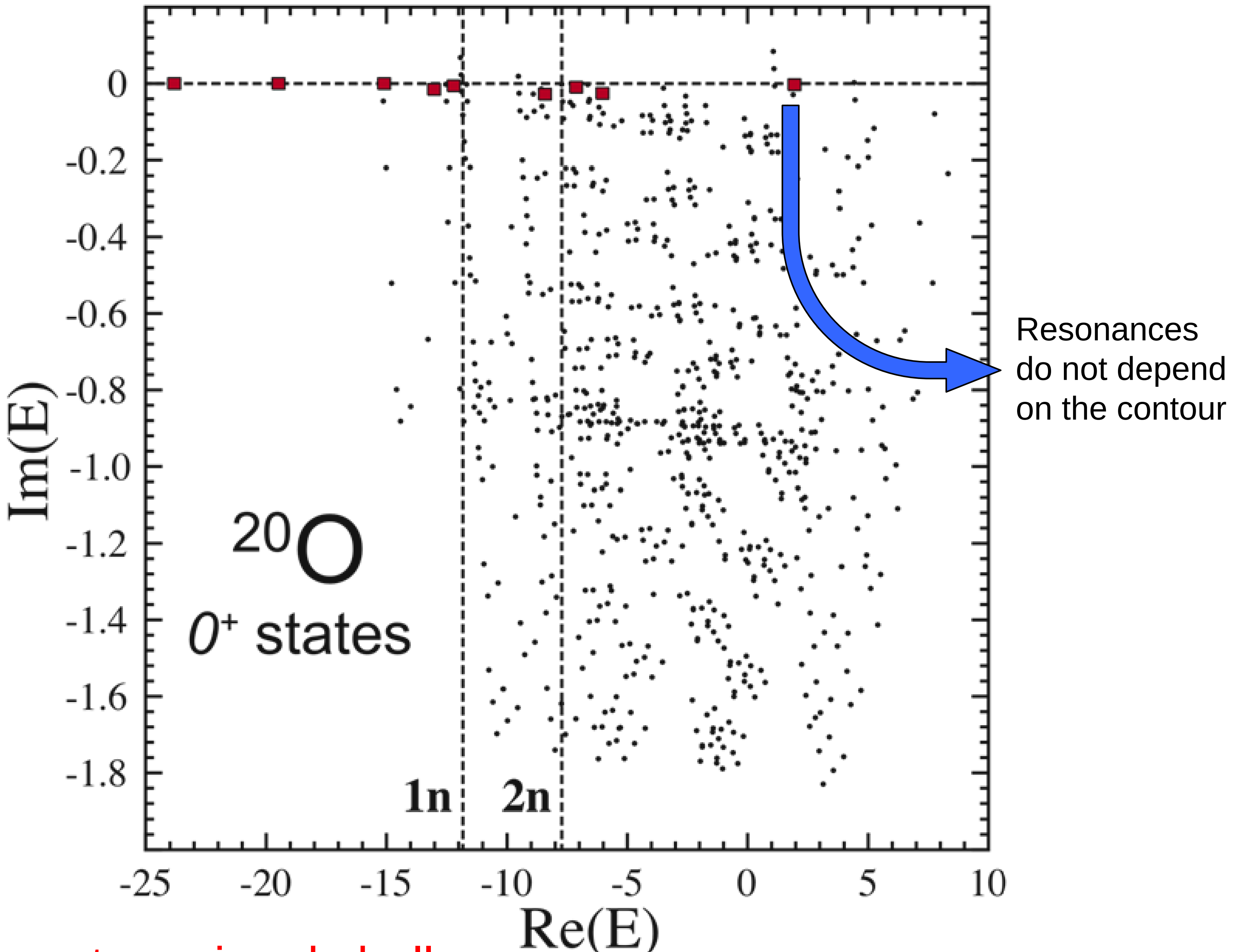
$$\sum_{n\in(b)} |u_n\rangle\langle u_n| + \int_0^{+\infty} |u(k)\rangle\langle u(k)|dk = 1.$$

Contour is discretized
Many-body Slater determinants (SD) are built

$$|SD_i\rangle = |u_{i_1}\cdots u_{i_A}\rangle \qquad \sum_i |SD_i\rangle\langle\widetilde{SD_i}| \simeq 1$$

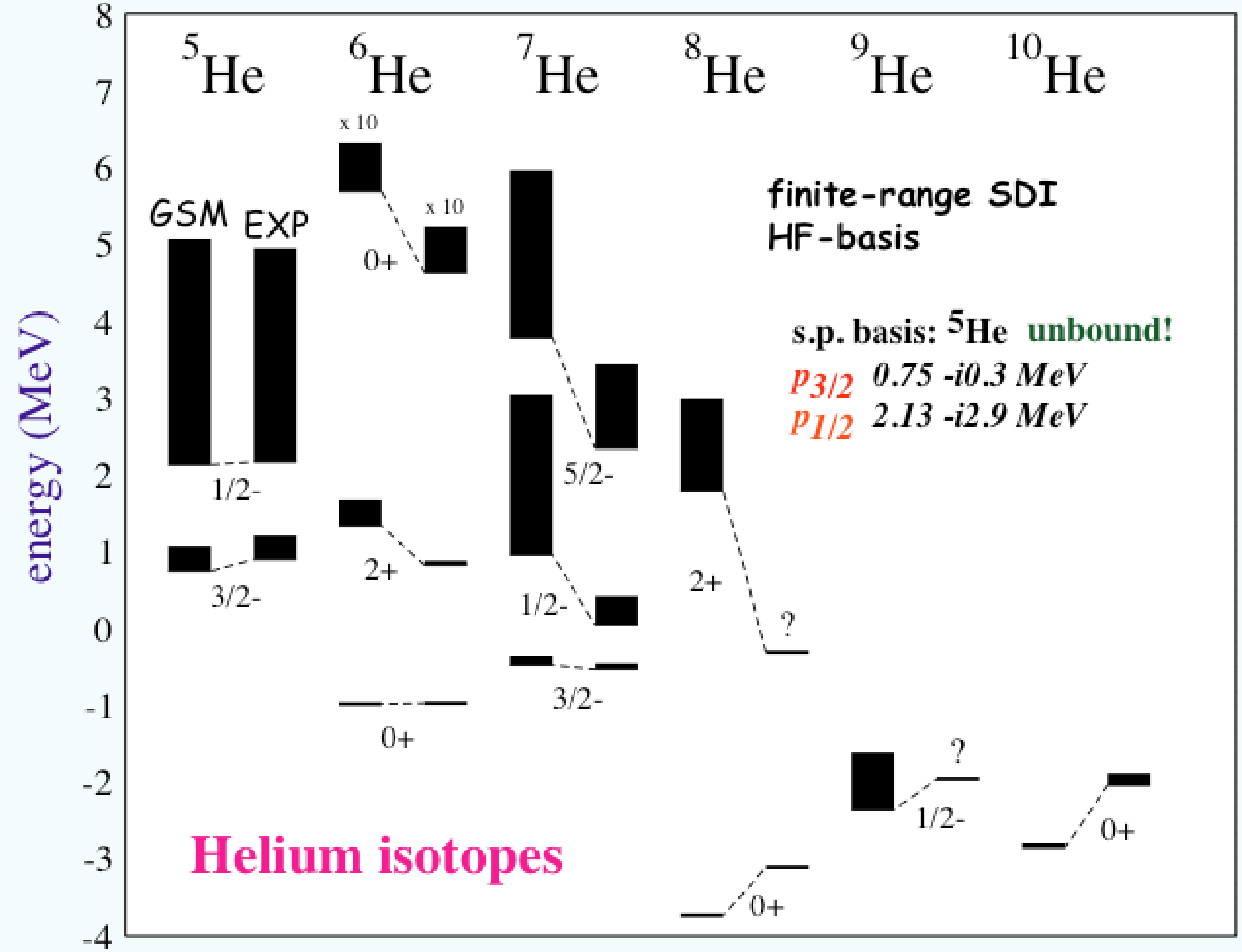
GSM Hamiltonian matrix is computed and diagonalized
(the matrix is complex symmetric!)

Selection of the many-body poles

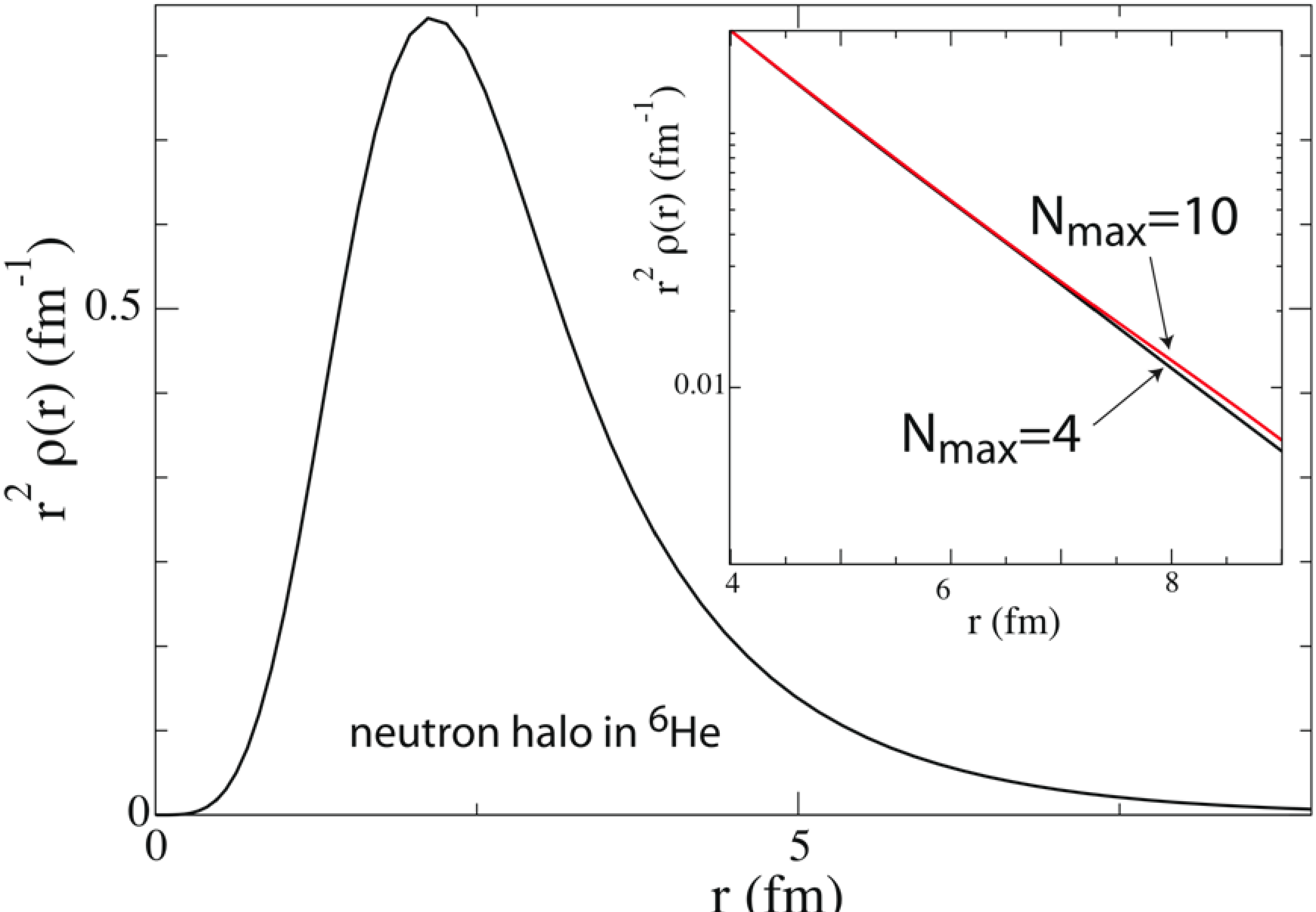


4 neutrons in sd shell

GSM: N. Michel et al., Phys.Rev.Lett. 89, 042502 (2002)



Neutron halo density



GSM+$V_{low\text{-}k}$: G. Hagen et al., Phys. Rev. C 73 (2006) 064307

Generalized Variational Principle

(a complex extension of the usual variational principle)

N. Moiseyev, P.R. Certain, and F. Weinhold, Mol. Phys. 36, 1613 (1978).
N. Moiseyev, Phys. Rep. 302, 212 (1998)

$$E[\Phi] = \frac{\langle \Phi^* | \hat{H} | \Phi \rangle}{\langle \Phi^* | \Phi \rangle}$$

is stationary around any eigenstate

$$\hat{H}|\Phi_0\rangle = E[\Phi_0]|\Phi_0\rangle$$

That is, $\delta_\Phi E[\Phi]_{\Phi=\Phi_0} = 0.$

It should be noted that the complex variational principle is a stationary principle rather than an upper of lower bound for either the real or imaginary part of the complex eigenvalue. However, it can be very useful when applied to the squared modulus of the complex eigenvalue. Indeed,

$$\delta_\Phi |E|^2 = \delta_\Phi (E^* E) = E^* \delta_\Phi E + E \delta_\Phi E^* = 0$$

Example: GSM+DMRG calculations for ^{7}Li

J. Rotureau et al., Phys. Rev. C 79, 014304 (2009)

$N_{\mathrm{opt.}}$	$\lvert E_{\max}\rvert$	$Re(E_{\max})$	$Im(E_{\max})$	$Re(E_{\mathrm{ave}})$	$Im(E_{\mathrm{ave}})$
40	22.6489	-22.6475	0.2470	-22.5270	0.2468
50	22.6605	-22.6591	0.2484	-22.61844	0.2478
60	22.6631	-22.6617	0.2485	-22.6510	0.2484
70	22.6634	-22.6620	0.2486	-22.6609	0.2486
80	22.6634	-22.6620	0.2486	-22.6619	0.2486

Threshold anomaly

E.P. Wigner, Phys. Rev. 73, 1002 (1948), the Wigner cusp
G. Breit, Phys. Rev. 107, 923 (1957)
A.I. Baz', JETP 33, 923 (1957)
A.I. Baz', Ya.B. Zel'dovich, and A.M. Perelomov, Scattering Reactions and Decay in Nonrelativistic Quantum Mechanics, Nauka 1966
A.M. Lane, Phys. Lett. 32B, 159 (1970)
S.N. Abramovich, B.Ya. Guzhovskii, and L.M. Lazarev, Part. and Nucl. 23, 305 (1992).

- The threshold is a branching point.
- The threshold effects originate in conservation of the flux.
- If a new channel opens, a redistribution of the flux in other open channels appears, i.e. a modification of their reaction cross-sections.
- The shape of the cusp depends strongly on the orbital angular momentum.

Y(b,a)X	$\sigma_\ell \sim k^{2\ell-1}$
X(a,b)Y	$\sigma_\ell \sim k^{2\ell+1}$

$$\begin{array}{lll} & & a+X \\ & & a_1+X_1 \text{ at } Q_1 \\ a+X & \longrightarrow & a_2+X_2 \text{ at } Q_2 \\ & & \\ & & a_n+X_n \text{ at } Q_n \end{array}$$

Threshold anomaly (cont.)

Studied experimentally and theoretically in various areas of physics:

pion-nucleus scattering

R.K. Adair, Phys. Rev. 111, 632 (1958)

A. Starostin et al., Phys. Rev. C 72, 015205 (2005)

electron-molecule scattering

W. Domcke, J. Phys. B 14, 4889 (1981)

electron-atom scattering

K.F. Scheibner et al., Phys. Rev. A 35, 4869 (1987)

ultracold atom-diatom scattering

R.C. Forrey et al., Phys. Rev. A 58, R2645 (1998)

Low-energy nuclear physics

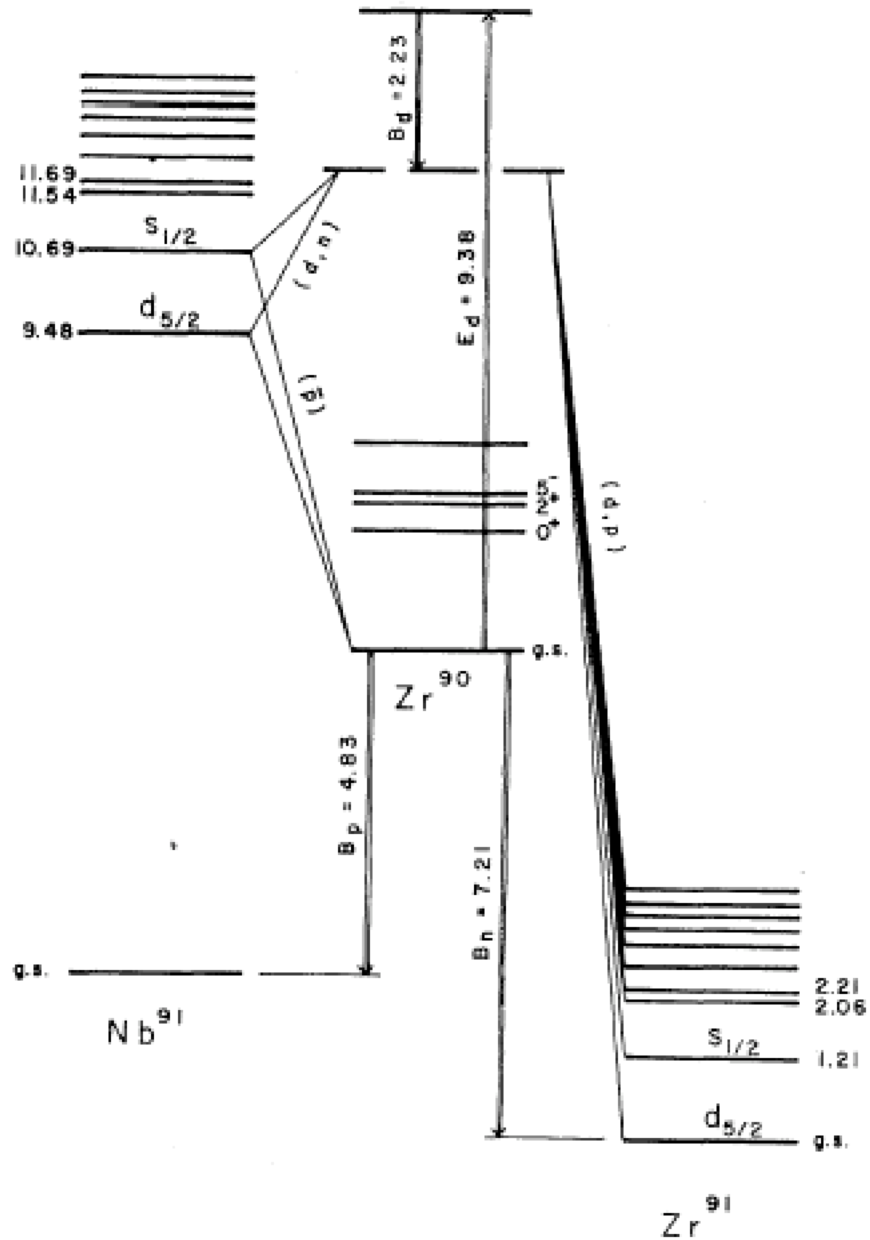
- charge-exchange reactions
- neutron elastic scattering
- deuteron stripping

The presence of cusp anomaly could provide structural information about reaction products. This is of particular interest for neutron-rich nuclei

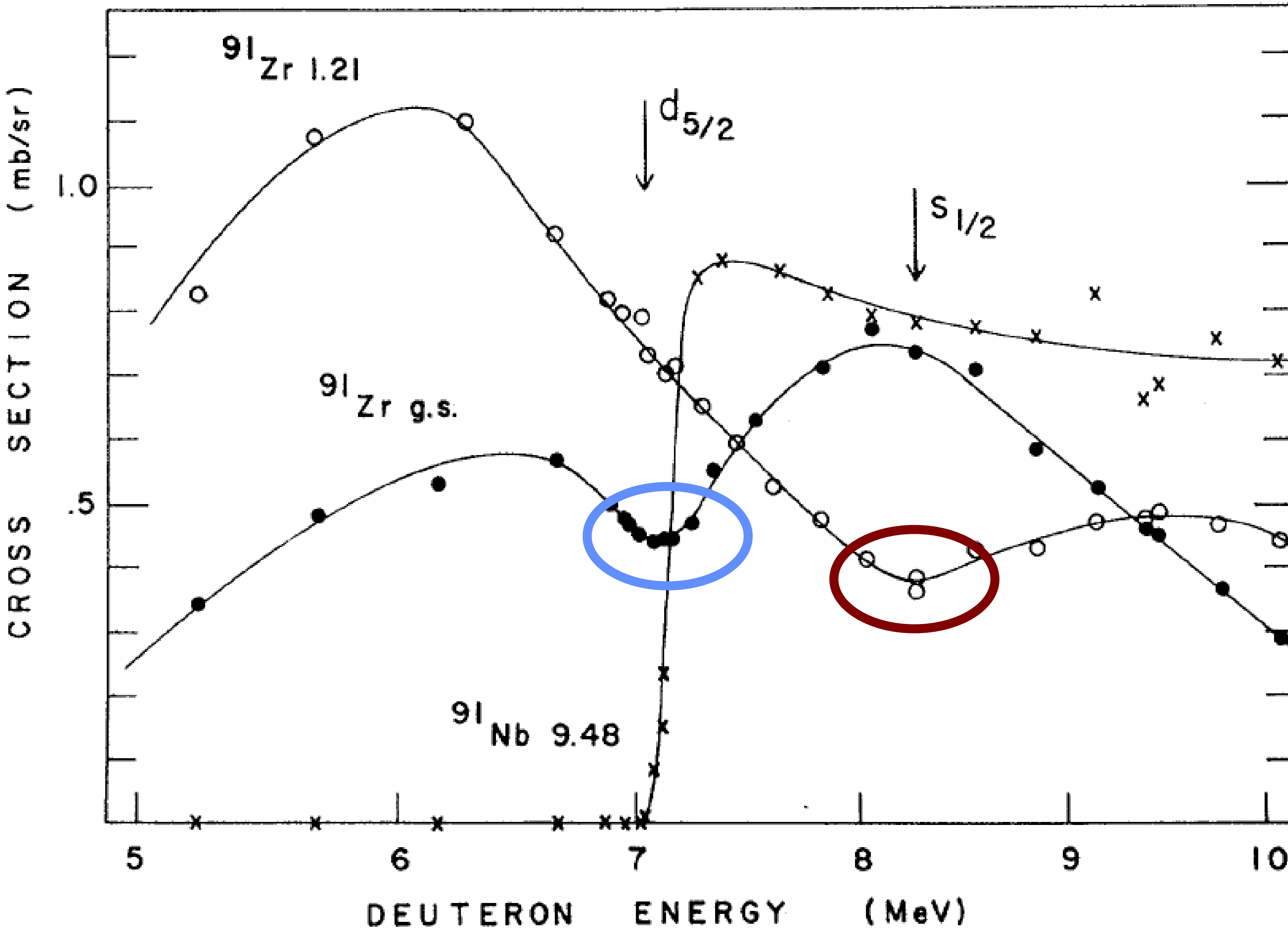
Coupling between analog states in (d,p) and (d,n)

C.F. Moore et al.
Phys. Rev. Lett. 17, 926 (1966)

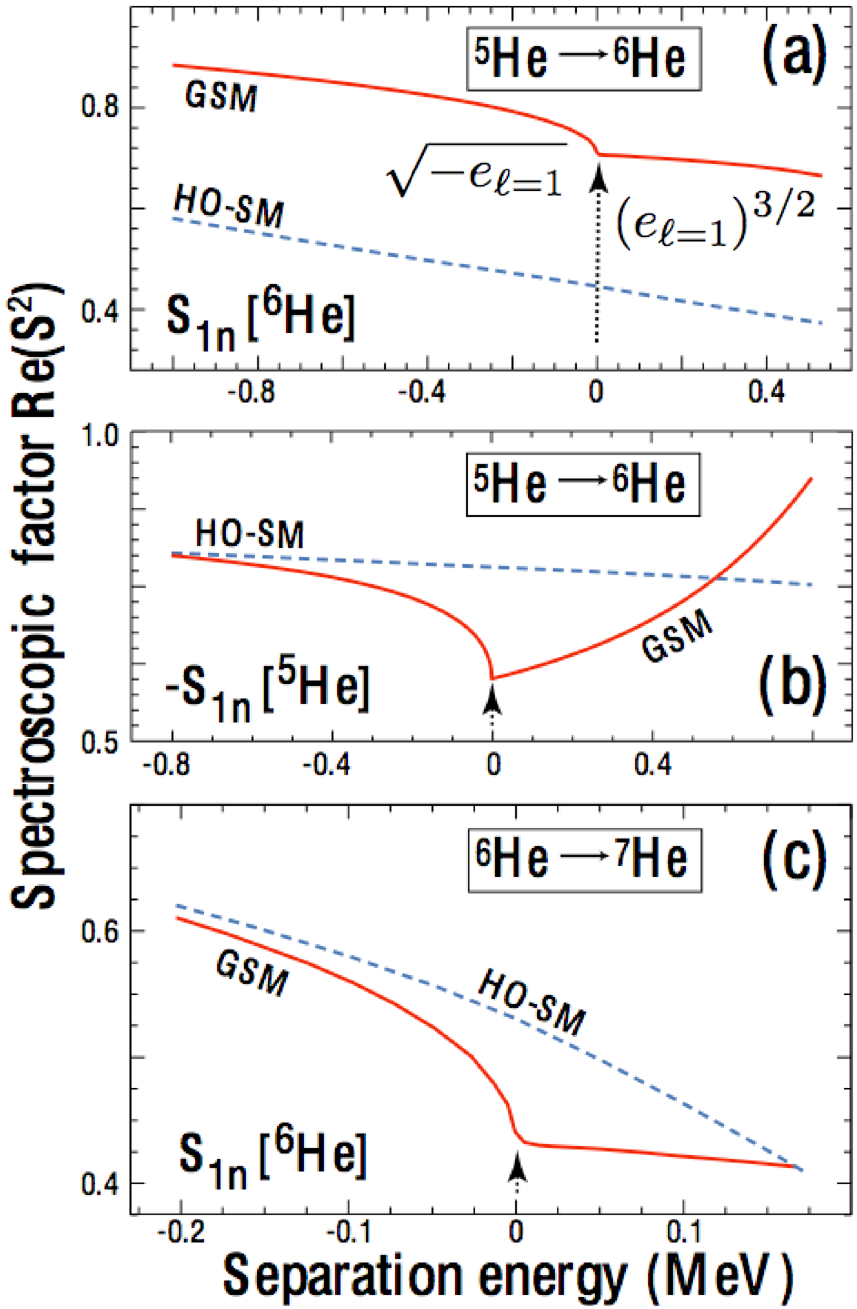
$$^{90}\mathrm{Zr} + d \rightarrow {}^{91}\mathrm{Zr} + p$$
$$^{90}\mathrm{Zr} + d \rightarrow {}^{91}\mathrm{Nb}^{A} + n$$
$$^{91}\mathrm{Nb}^{A} \rightarrow {}^{90}\mathrm{Zr} + p$$



C.F. Moore et al., Phys. Rev. Lett. 17, 926 (1966)



N. Michel et al. PRC 75, 0311301(R) (2007)



WS potential depth decreased to bind ^{7}He. Monopole SGI strength varied

^{5}He+n $\longrightarrow$ ^{6}He

Overlap integral, basis independent!

^{6}He+n $\longrightarrow$ ^{7}He

Anomalies appear at calculated thresholds (many-body S-matrix unitary)

Scattering continuum essential

see also Nucl. Phys. A 794, 29 (2007)

Many-body OQS calculations correctly predict the Wigner-cusp and channel-coupling threshold effects. This constitutes a very strong theoretical check for the GSM approach.

The spectroscopic factors defined in the OQS framework through the norm of the overlap integral, exhibit strong variations around particle thresholds. Such variations cannot be described in a standard CQS SM framework that applies a "one-isolated-state" ansatz and ignores the coupling to the decay and scattering channels. In the GSM model calculations, the contribution to SF from a non-resonant continuum can be as large as 25%.

- The non-resonant continuum is important for the spectroscopy of weakly bound nuclei (energy shifts of excited states, additional binding,…)
- SFs, cross sections, etc., exhibit a non-perturbative and non-analytic behavior (cusp effects) close to the particle-emission thresholds. These anomalies strongly depend on orbital angular momentum
- Microscopic CSM (GSM) fully accounts for channel coupling

Timofeyuk, Blokhintsev, Tostevin, Phys. Rev. C68, 021601 (2003)

Non-Borromean two-neutron halos

TABLE I. Non-Borromean two-nucleon halo nuclei A, one-nucleon halo nuclei $A-1$, and their common $A-2$-body cores. The two- and one-nucleon thresholds $S_A(2N)$, $S_A(1N)$, and $S_{A2\ 1}(1N)$ are also shown.

A	$A-1$	$A-2$	$S_A(2N)$ MeV	$S_A(1N)$ MeV	$S_{A-1}(1N)$ MeV
^{12}Be	^{11}Be($\frac{1}{2}^+$)	^{10}Be	3.670	3.170	0.500
^{12}Be	^{11}Be($\frac{1}{2}^-$)	^{10}Be	3.670	3.490	0.180
^{15}B	^{14}B(1^-)	^{13}B	3.740	3.510	0.230
^{9}C	^{8}B(g.s.)	^{7}Be	1.433	1.296	0.137
^{16}C	^{15}C($\frac{1}{2}^+$)	^{14}C	5.469	4.251	1.218
^{16}C	^{15}C($\frac{5}{2}^+$)	^{14}C	5.469	4.991	0.478

Complex Scaling

Introduced in the early 1970s in atomic physics to guarantee that wave functions and resonances are square integrable.

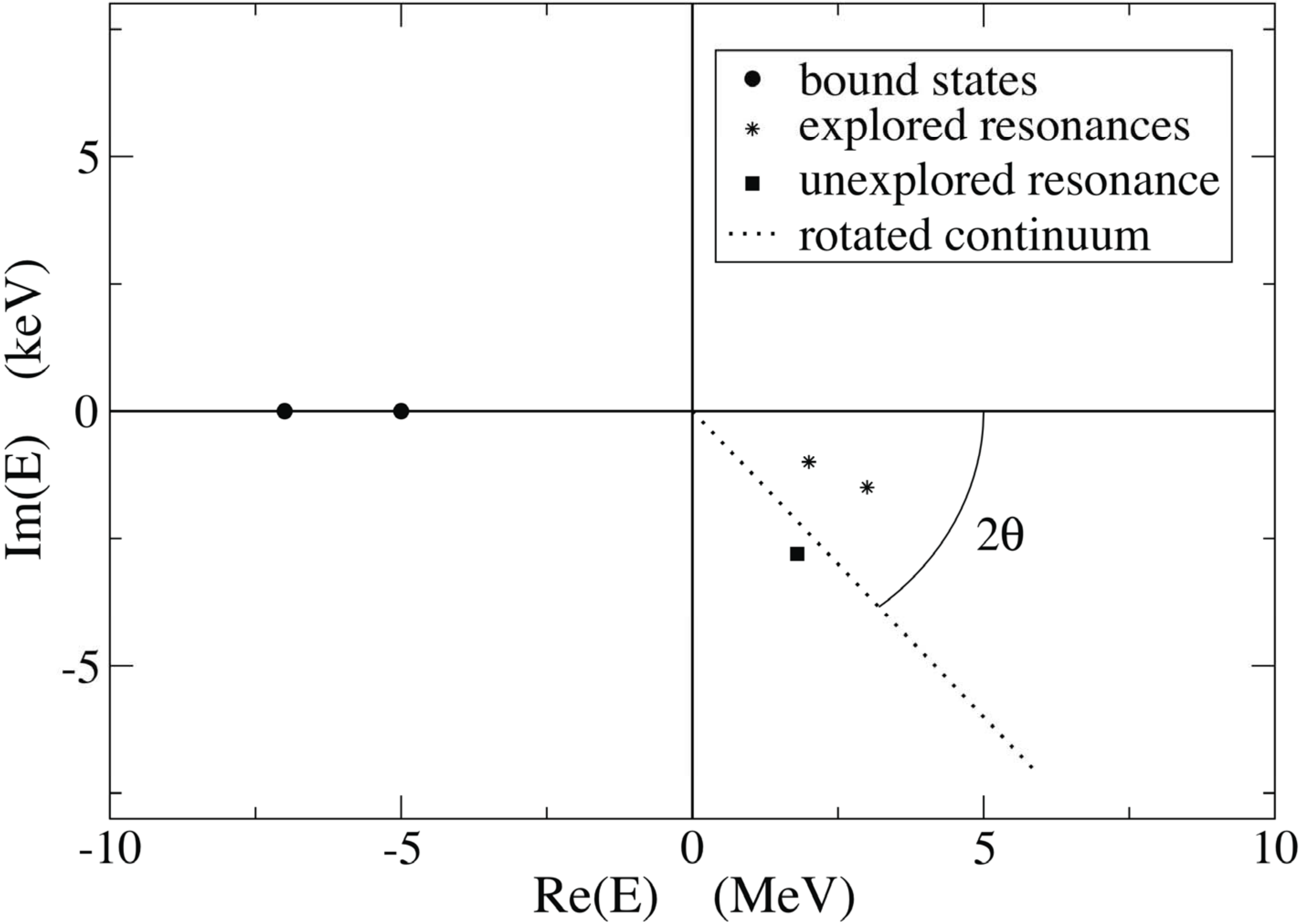
$$\mathbf{r} \to e^{i\theta}\mathbf{r}$$

$$\hat{U}(\theta)\psi(\mathbf{r}) = e^{i\frac{3}{2}\theta}\psi(\mathbf{r}e^{i\theta})$$

Uniform complex scaling

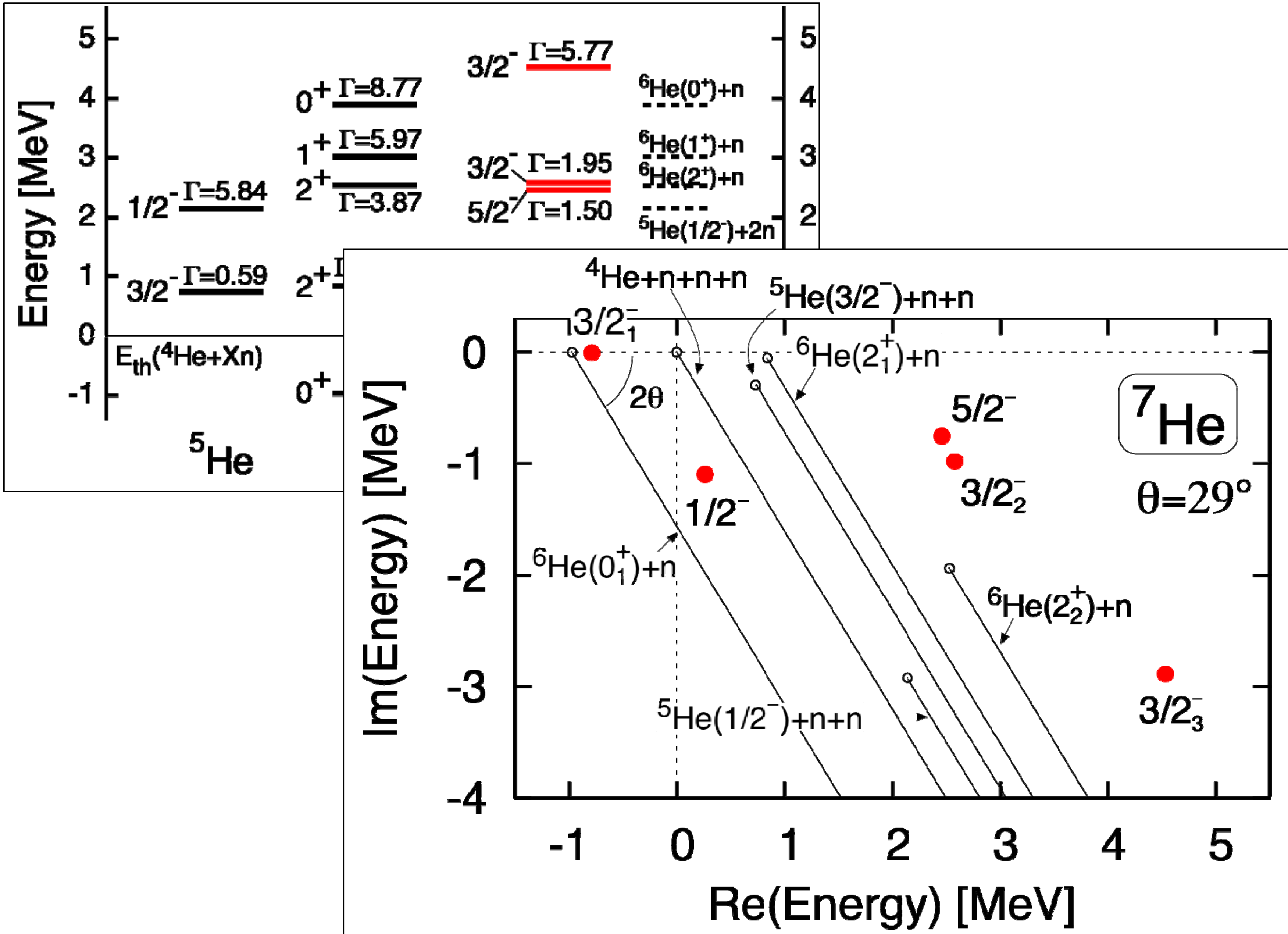
The transformed Hamiltonian is no longer hermitian as it acquires a complex potential. However, for a wide class of local and nonlocal potentials, called *dilation-analytic potentials*, the so-called ABC is valid:

- The bound states of h and h_θ are the same;
- The positive-energy spectrum of the original Hamiltonian h is rotated down by an angle of 2θ into the complex-energy plane;
- The resonant states of h with eigenvalues E_n satisfying the condition $|\arg(E_n)|<2\theta$ are also eigenvalues h_θ and their wave functions are square integrable.

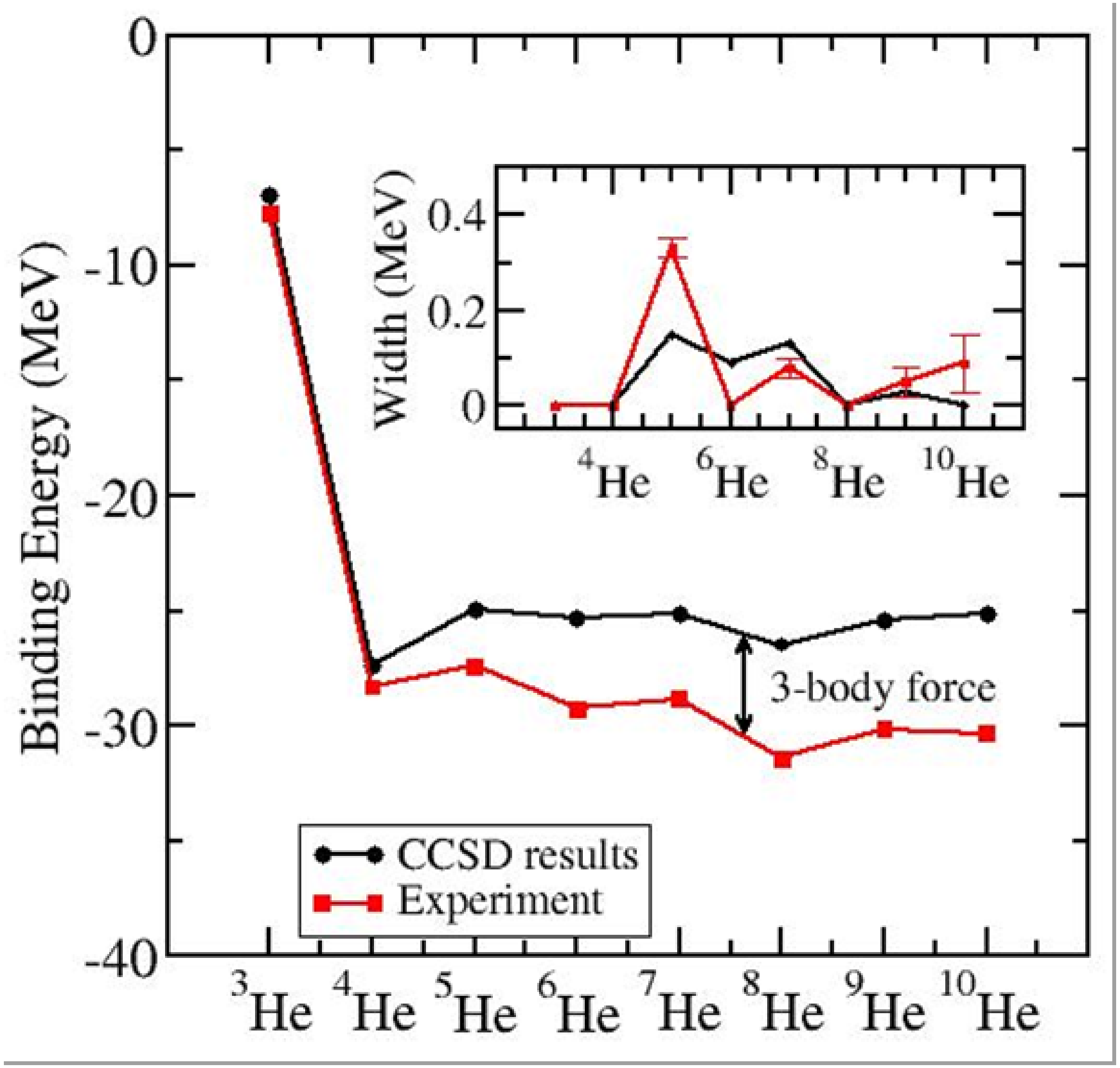


bound states
explored resonances
unexplored resonance
rotated continuum
2θ
5
0
-5
-10
-5
0
5
10
Im(E) (keV)
Re(E) (MeV)

Myo, Kato, Ikeda, PRC C76, 054309 (2007)



Helium resonances in the Coupled Cluster approach and Berggren basis



Complex coupled-cluster approach to an ab-initio description of open quantum systems, *G. Hagen, D. J. Dean, M. Hjorth-Jensen and T. Papenbrock, Phys. Lett. B656, 169 (2007)*