

Equivalent Hamiltonians for PT-Symmetric Versions of Dual 2-D Field Theories

H. F. Jones
Imperial College London

Outline

1. Systematics of Commutation Relations for Q , h
2. Q and h for PT-Symmetric Versions of Sine-Gordon and Massive Thirring Models[†]
3. How it works
4. Conclusions

[†]C. M. Bender, HFJ and R. J. Rivers, Phys. Lett. **B625** (2005) 333

1. Commutation Relations

Talking about Hamiltonians that are not Hermitian, but are:

- (i) PT-symmetric
- (ii) pseudo-Hermitian

i.e.

$$H^\dagger = \eta H \eta^{-1} \quad (1.1)$$

Here η is **Hermitian** and **positive definite**, and is usefully written as

$$\eta = e^{-Q}$$

where Q is Hermitian.

Specialize to case where $H = H_0 + \varepsilon H_1$, where

- H_0 is Hermitian: $H_0^\dagger = H_0$
- H_1 is anti-Hermitian: $H_1^\dagger = -H_1$

Looking for a perturbative solution: $Q = \sum_r Q_r \varepsilon^r$ (r odd)

(i) Commutation relations for Q

In first place Eq. (1.1) becomes

$$\begin{aligned} H^\dagger &= e^{-Q} H e^Q \\ &= H + [H, Q] + \frac{1}{2!} [[H, Q], Q] + \frac{1}{3!} [[[H, Q], Q], Q] \\ &\quad + \dots + \frac{1}{n!} \underbrace{[\dots [H, Q], \dots, Q]}_{n \text{ commutators}} + \dots \end{aligned}$$

Now insert $H = H_0 + \varepsilon H_1$, $Q = \sum_r Q_r \varepsilon^r$ and collect terms:

$$-2H_1 = [H_0, Q_1]$$

$$0 = [H_0, Q_3] + \frac{1}{2!} [[H_1, Q_1], Q_1] + \frac{1}{3!} [[[H_0, Q_1], Q_1], Q_1]$$

$$0 = [H_0, Q_5] + \frac{1}{2!} ([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1])$$

$$+ \frac{1}{3!} ([[[H_0, Q_1], Q_1], Q_3] + \textit{perms})$$

$$+ \frac{1}{4!} ([[[[H_1, Q_1], Q_1], Q_1], Q_1])$$

$$+ \frac{1}{5!} ([[[[[H_0, Q_1], Q_1], Q_1], Q_1], Q_1])$$

.....

Here coefficients are simple: just $1/n!$. But in each equation we can use previous equations to eliminate $[H_0, Q_r]$. They then become

$$[H_0, Q_1] = -2H_1,$$

$$[H_0, Q_3] = -\frac{1}{6}[[H_1, Q_1], Q_1],$$

$$[H_0, Q_5] = -\frac{1}{6}([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) \\ + \frac{1}{360}[[[[H_1, Q_1], Q_1], Q_1], Q_1]$$

$$\begin{aligned}
[H_0, Q_7] &= -\frac{1}{6}[[H_1, Q_3], Q_3] \\
&\quad -\frac{1}{6}([[H_1, Q_1], Q_5] + [[H_1, Q_5], Q_1]) \\
&\quad +\frac{1}{360}([[[[H_1, Q_1], Q_1], Q_1], Q_3] + perms) \\
&\quad -\frac{1}{15120}([[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1], Q_1] \\
&\quad \dots\dots\dots
\end{aligned}$$

Q1: What are these coefficients, and what is general coefficient?

Well, c_n , coefficient of $2n$ -fold commutator, is

$$\begin{aligned}c_n &= -\frac{2B_{2n}}{(2n)!}, \\ &= \text{coefficient of } z^{2n} \text{ in } -z \coth \frac{1}{2}z\end{aligned}$$

Q1: What are these coefficients, and what is general coefficient?

Well, c_n , coefficient of $2n$ -fold commutator, is

$$\begin{aligned}c_n &= -\frac{2B_{2n}}{(2n)!}, \\ &= \text{coefficient of } z^{2n} \text{ in } -z \coth \frac{1}{2}z\end{aligned}$$

Q2: Why?

Well, \exists a recursion relation for c_n :

$$c_n = -\frac{1}{(2n)!} - \underbrace{\sum_{r=1}^n \frac{c_{n-r}}{(2r+1)!}}_{\text{from } H_0 \text{ comm}^s} \quad (1.2)$$

from H_0 comm^s

- same as one of recursion relations for c_n in

$$-z \coth \frac{1}{2}z = \sum_{n=0}^{\infty} c_n z^{2n}.$$

Multiply both sides by $e^{z/2}$:

$$-z(e^z + 1) = \sum_{n=0}^{\infty} c_n z^{2n} (e^z - 1),$$

i.e.

$$-z \left(1 + \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) = \sum_{m=0}^{\infty} c_m z^{2m} \sum_{s=1}^{\infty} \frac{z^s}{s!}$$

Equating coefficients of z^{2n+1} gives precisely Eq. (1.2).

Commutation relations for h

Equivalent Hermitian Hamiltonian is

$$\begin{aligned}
 h &= e^{-Q/2} H e^{Q/2} \\
 &= H + \frac{1}{2} [H, Q] + \frac{1}{4 \times 2!} [[H, Q], Q] + \frac{1}{8 \times 3!} [[[H, Q], Q], Q] \\
 &\quad + \cdots + \frac{1}{2^n \times n!} \underbrace{[\dots [H, Q], \dots, Q]}_{n \text{ commutators}} + \dots
 \end{aligned} \tag{1.3}$$

Now insert $H = H_0 + \varepsilon H_1$, $Q = \sum_r Q_r \varepsilon^r$, $h = \sum_{r \text{ even}} h_r \varepsilon^r$:

First few equations are

$$h_0 = H_0$$

$$h_2 = \frac{1}{2}[H_1, Q_1] + \frac{1}{4 \times 2!}[[H_0, Q_1], Q_1]$$

$$h_4 = \frac{1}{2}[H_1, Q_3] + \frac{1}{4 \times 2!}([[H_0, Q_1], Q_3] + [[H_0, Q_3], Q_1]) \\ + \frac{1}{8 \times 3!}[[[H_1, Q_1], Q_1], Q_1] \\ + \frac{1}{16 \times 4!}[[[[H_0, Q_1], Q_1], Q_1], Q_1]$$

$$\begin{aligned}
h_6 = & \frac{1}{2}[H_1, Q_5] + \frac{1}{4 \times 2!}([H_0, Q_1], Q_5 + [[H_0, Q_5], Q_1]) \\
& + \frac{1}{4 \times 2!}[[H_0, Q_3], Q_3] \\
& + \frac{1}{8 \times 3!}([[[H_1, Q_1], Q_1], Q_3] + perms) \\
& + \frac{1}{16 \times 4!}([[[[H_0, Q_1], Q_1], Q_3], Q_1] + perms) \\
& + \frac{1}{32 \times 5!}([[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1] \\
& + \frac{1}{64 \times 6!}([[[[[[H_0, Q_1], Q_1], Q_1], Q_1], Q_1], Q_1]
\end{aligned}$$

Again, coefficients are simple: just $1/(2^n n!)$. But eliminating $[H_0, Q_r]$ gives

$$h_0 = H_0$$

$$h_2 = \frac{1}{4}[H_1, Q_1]$$

$$h_4 = \frac{1}{4}[H_1, Q_3] - \frac{1}{192}[[[H_1, Q_1], Q_1], Q_1]$$

$$h_6 = \frac{1}{4}[H_1, Q_5] - \frac{1}{192}([[[H_1, Q_1], Q_1], Q_3] + perms) \\ + \frac{1}{7680}[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1]$$

.....

Q3: What are these coefficients, and what is general coefficient?

Well, a_n , coefficient of $(2n - 1)$ -fold commutator, is

$$\begin{aligned} a_n &= -\frac{E_{2n-1}(0)}{2^{2n-1} \times (2n - 1)!}, \\ &= \text{coefficient of } z^{2n-1} \text{ in } \tanh(z/4) \end{aligned}$$

Q3: What are these coefficients, and what is general coefficient?

Well, a_n , coefficient of $(2n - 1)$ -fold commutator, is

$$\begin{aligned} a_n &= -\frac{E_{2n-1}(0)}{2^{2n-1} \times (2n - 1)!}, \\ &= \text{coefficient of } z^{2n-1} \text{ in } \tanh(z/4) \end{aligned}$$

Q4: Why?

Well, recursion relation for a_n is:

$$a_n = \frac{1}{2^{2n-1} \times (2n-1)!} + \underbrace{\sum_{r=1}^n \frac{c_{n-r}}{(2^{2r} \times (2r)!)} }_{\text{from } H_0 \text{ comm}^s} \quad (1.4)$$

Can obtain same recursion relation for coefficients in

$$\tanh(z/4) = \sum_{n=1}^{\infty} a_n z^{2n-1}$$

Thus

$$z \sinh(z/2) = \sum_{n=1}^{\infty} \frac{z^{2n}}{2^{2n-1} \times (2n-1)!}$$

and

$$\begin{aligned} (z \coth(z/2))(\cosh(z/2) - 1) &= - \left(\sum_{m=0}^{\infty} c_m z^{2m} \right) \left(\sum_{r=1}^{\infty} \frac{z^{2r}}{(2^{2r} \times (2r)!)} \right) \\ &= - \sum_{n=1}^{\infty} z^{2n} \sum_{r=1}^n \frac{c_{n-r}}{(2^{2r} \times (2r)!)} \end{aligned}$$

Subtracting gives

$$\begin{aligned} \sum_{n=1}^{\infty} a_n z^{2n} &= z[\sinh(z/2) - \coth(z/2)(\cosh(z/2) - 1)] \\ &= z \tanh(z/4) ! \end{aligned}$$

2. PT-Symmetric Versions of Sine-Gordon and Massive Thirring Models

(i) Modified Sine-Gordon model

Conventional S-G model, in [1+1], has

$$\mathcal{L}_{\text{SG}} = \frac{1}{2}(\partial_{\mu}\varphi)^2 + \frac{M^2}{\lambda^2}(\cos \lambda\varphi - 1),$$

or correspondingly

$$\mathcal{H}_{\text{SG}} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{\lambda^2}(1 - \cos \lambda\varphi).$$

Modified version (*PT*-symmetric, but not Hermitian):

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{\lambda^2}(1 - \cos \lambda\varphi - i\varepsilon \sin \lambda\varphi),$$

So $\mathcal{H}_0 = \mathcal{H}_{SG}$, and $\mathcal{H}_1 = -i(M^2/\lambda^2) \sin \lambda\varphi$.

First few equations for Q_n are:

$$\begin{aligned} [H_0, Q_1] &= -2H_1, \\ [H_0, Q_3] &= -\frac{1}{6} [[H_1, Q_1], Q_1], \\ [H_0, Q_5] &= -\frac{1}{6} ([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) \\ &\quad + \frac{1}{360} [[[[H_1, Q_1], Q_1], Q_1], Q_1] \end{aligned} \tag{2.1}$$

Ansatz for Q_1 :

$$Q_1 = \xi_1 \int_x \Pi_x.$$

Then

$$\begin{aligned} [H_0, Q_1] &= \xi_1 \int_{xy} \left[\frac{1}{2} \partial_1 \varphi_x \partial_1 \varphi_x - \frac{M^2}{\lambda^2} \cos \lambda \varphi, \Pi_y \right] \\ &= i \xi_1 \int_x \left(\underbrace{\partial_1^2 \varphi_x}_{\int \rightarrow 0} + \underbrace{\frac{M^2}{\lambda} \sin \lambda \varphi}_{\propto \mathcal{H}_1} \right), \end{aligned}$$

$$\because [\varphi_x, \Pi_y] = i \delta_{xy}$$

So $[H_0, Q_1] = -2H_1$, if $\xi_1 = 2/\lambda$

Similarly set $Q_3 = \xi_3 \int_x \Pi_x$

Then (2.1) $\Rightarrow \xi_3 = \xi_1/3$

Similarly

$$\xi_5 = \xi_1/5$$

Seem to be generating series for $\tanh^{-1} \varepsilon$, giving all-orders result

$$Q = \frac{2\delta}{\lambda} \int_x \Pi_x$$

where $\delta = \tanh^{-1} \varepsilon$

N.B. Only makes sense for $|\varepsilon| < 1$.

Can verify result *a posteriori* by constructing

$$h = e^{-(\delta/\lambda) \int_x \Pi_x} H e^{(\delta/\lambda) \int_x \Pi_x}$$

This just shifts φ : $\varphi \rightarrow \varphi + i\delta/\lambda$.

Thus

$$\begin{aligned}\cos \lambda\varphi + i\varepsilon \sin \lambda\varphi &\equiv \operatorname{sech}\delta \cos(\lambda\varphi - i\delta) \\ &\rightarrow \operatorname{sech}\delta \cos \lambda\varphi\end{aligned}$$

$\therefore h$ is again S-G model, but with bare mass M' given by

$$(M')^2 = M^2 \operatorname{sech}\delta = M^2 (1 - \varepsilon^2)^{\frac{1}{2}} \quad (2.2)$$

Now can understand restriction $|\varepsilon| < 1$: PT spontaneously broken for $|\varepsilon| > 1$.

(ii) Modified massive Thirring model

Conventional MT model, in [1+1], has

$$\mathcal{L}_{\text{MT}} = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi),$$

or correspondingly

$$\mathcal{H}_{\text{MT}} = \bar{\psi}(-i\not{\nabla} + m)\psi - \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi).$$

This is equivalent to S-G model, with correspondence

$$\frac{\lambda^2}{4\pi} = \frac{1}{1 - g/\pi}, \quad M^2 = m\Lambda \quad (\Lambda = \text{ren. scale})$$

In particular $\lambda = \sqrt{4\pi} \leftrightarrow g = 0$, the free fermion theory.

Modified version involves a “ γ_5 - dependent mass”:

$$\mathcal{H} = \bar{\psi}(-i\not{D} + m(1 + \varepsilon\gamma_5))\psi - \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$$

Here $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_2$, $\gamma_5 \equiv \gamma_0\gamma_1 = -\sigma_3$.

Additional term is PT-symmetric but non-Hermitian.

First consider $g = 0$ ($\lambda = \sqrt{4\pi}$).

Write

$$Q_1 = \int_{xy} \psi_x^\dagger (G_1)_{xy} \psi_y$$
$$H_0 = \int_{xy} \psi_x^\dagger D_{xy} \psi_y,$$

where $D = \gamma_0(-i\cancel{\nabla} + m) = -i\gamma_5\partial_1 + m\gamma_0$.

Using canonical equal-time anti-commⁿ relⁿ $\{\psi_x^\dagger, \psi_y\} = \delta_{xy}$, eqⁿ

$$[H_0, Q_1] = -2H_1,$$

reads

$$\begin{aligned} -2m \int \bar{\psi}\gamma_5\psi &= \int [\psi^\dagger D\psi, \psi^\dagger G_1\psi] \\ &= \int \psi^\dagger [D, G_1]\psi \\ &= \int \psi^\dagger [-i\gamma_5\partial_1 + \underline{m\gamma_0}, G_1]\psi \end{aligned}$$

Particular solution is $G_1 = -\gamma_5$

Similarly set $G_3 = -\xi_3 \gamma_5$.

Then (2.1) \Rightarrow $\xi_3 = 1/3$

Similarly third eqⁿ gives $\xi_5 = 1/5$

Again, seem to be generating series for $\tanh^{-1} \varepsilon$, with all-orders result

$$Q = -\delta \int_x (\psi^\dagger \gamma_5 \psi)_x,$$

where again $\delta = \tanh^{-1} \varepsilon$

Check result by constructing

$$h = \exp\left(\frac{1}{2}\delta \int_x (\psi^\dagger \gamma_5 \psi)_x\right) H \exp\left(-\frac{1}{2}\delta \int_x (\psi^\dagger \gamma_5 \psi)_x\right)$$

By virtue of Lorentz-like commⁿ relations

$$[\gamma_5, \gamma_0] = 2\gamma_1,$$

$$[\gamma_5, \gamma_1] = 2\gamma_0,$$

this is just

$$h = \bar{\psi}(-i\not{X} + \mu)\psi$$

where $\mu = m \operatorname{sech}\delta = m(1 - \varepsilon^2)^{\frac{1}{2}}$, in agreement with (2.2).

N.B. This Q also works for $g \neq 0$, since

$$(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) = (\psi^\dagger\psi)^2 - (\psi^\dagger\gamma_5\psi)^2.$$

Each term on RHS commutes with $Q = -\delta \int_x (\psi^\dagger \gamma_5 \psi)_x$.

Thus only effect on H is to change γ_5 -dependent mass term $m\bar{\psi}(1 + \varepsilon\gamma_5)\psi$ to a normal mass term $\mu\bar{\psi}\psi$.

Again, if $|\varepsilon| > 1$, PT symmetry is broken.

Another Solution

As usual, Q is not unique. Another possible (formal) solution is

$$G = i\gamma_5 \tanh^{-1} \left(\frac{\varepsilon m}{\not{\nabla}} \right) !$$

This mixes kinetic term and γ_5 mass term.

Corresponding h (for $g = 0$) is

$$h = \bar{\psi} \left[m - i\not{\nabla} \left(1 + \frac{\varepsilon^2 m^2}{\nabla^2} \right)^{\frac{1}{2}} \right] \psi$$

Strange way of writing free field theory with mass μ !
Equation of motion is

$$\left[\partial_0^2 - \nabla^2 \left(1 + \frac{\varepsilon^2 m^2}{\nabla^2} \right) \right] \psi = -m^2 \psi$$

i.e.

$$\partial^2 \psi = - \underbrace{m^2 (1 - \varepsilon^2)}_{\mu^2} \psi !$$

This Q only defined in a limited range of p -space.

3. How does this work?

(i) Where does \tanh^{-1} come from?

Important feature here: $Q_r = \alpha_r Q_1$ - all with same structure -
so that $Q = (\sum_r^\infty \alpha_r \varepsilon^r) Q_1$.

∴ from

$$[H_0, Q_1] = c_0 H_1,$$

$$[H_0, Q_3] = c_1 [[H_1, Q_1], Q_1],$$

get

$$[[H_1, Q_1], Q_1] = \alpha_3 \frac{c_0}{c_1} H_1 = \underline{4H_1}$$

$$(\alpha_3 = 1/3, c_0 = -2, c_1 = -1/6)$$

Then eqⁿ

$$\begin{aligned} [H_0, Q_5] &= c_1 ([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) \\ &\quad + c_2 [[[[H_1, Q_1], Q_1], Q_1], Q_1] \end{aligned}$$

becomes

$$-2\alpha_5 H_1 = [2(4c_1)\alpha_1\alpha_3 + (16c_2)\alpha_1^4] H_1$$

$$\Rightarrow \alpha_5 = 1/5$$

In fact all equations can be generated by expansion of

$$-2 \sum_{r \text{ odd}}^{\infty} \alpha_r \varepsilon^r = \varepsilon \sum_{r=0}^{\infty} \underbrace{4^r c_r}_{\hat{c}_r} (\alpha_1 \varepsilon + \alpha_3 \varepsilon^3 + \alpha_5 \varepsilon^5 + \dots)^{2r} \quad (3.1)$$

namely

$$\begin{aligned} -2\alpha_1 &= \hat{c}_0 \\ -2\alpha_3 &= \hat{c}_1\alpha_1^2 \\ -2\alpha_5 &= 2\hat{c}_1\alpha_1\alpha_3 + \hat{c}_2\alpha_1^4 \\ -2\alpha_7 &= \hat{c}_1(2\alpha_1\alpha_5 + \alpha_3^2) + 4\hat{c}_2\alpha_1^3\alpha_3 + \hat{c}_3\alpha_1^6 \\ &\dots\dots \end{aligned}$$

Multiplicities are no. of perm^{ns} of a given commutator structure.

Then, assuming that α_r are indeed coefficients of $\operatorname{arctanh}$,
Eq. (3.1) reads

$$\begin{aligned} -2\delta &= \varepsilon \sum_r^{\infty} c_r (2\delta)^{2r} \\ &= \tanh \delta (-2\delta \coth \delta) \quad \checkmark \checkmark \checkmark \end{aligned}$$

(ii) Where does sech come from?

Well, with $[[H_1, Q_1], Q_1] = 4H_1$, $[H_1, Q_1] = -2H_0$, eq^s for h :

$$h_2 = a_1[H_1, Q_1]$$

$$h_4 = a_1[H_1, Q_3] + a_2[[[H_1, Q_1], Q_1], Q_1],$$

$$h_6 = a_1[H_1, Q_5] + a_2([[[H_1, Q_1], Q_1], Q_3] + perms) \\ + a_3([[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1),$$

become

$$h_2 = -2\hat{a}_1\alpha_1H_0$$

$$h_4 = -2(\hat{a}_1\alpha_3 + \hat{a}_2\alpha_1^3)H_0$$

$$h_6 = -2(\hat{a}_1\alpha_5 + 3\hat{a}_2\alpha_1^2\alpha_3 + \hat{a}_3\alpha_1^5)H_0$$

where $\hat{a}_r = 4^{r-1}a_r$.

Again, $h \equiv \sum h_r \varepsilon^r$ is generated by

$$\begin{aligned} & 1 - 2\varepsilon \sum_{r=1}^{\infty} \hat{a}_r (\alpha_1 \varepsilon + \alpha_3 \varepsilon^3 + \alpha_5 \varepsilon^5 + \dots)^{2r-1} \\ = & 1 - \varepsilon \sum_{r=1}^{\infty} a_r (2\delta)^{2r-1} \\ = & 1 - \tanh \delta \times \tanh(\delta/2) \\ = & \operatorname{sech} \delta !! \end{aligned}$$

4. Conclusions

- (i) Have identified the coefficients of multiple commutators involved in perturbative expansions for Q and h :

$$H^\dagger = e^{-Q} H e^Q$$
$$h = e^{-\frac{1}{2}Q} H e^{\frac{1}{2}Q}$$

- (ii) Have found exact expressions for Q and h in PT-symmetric versions of Sine-Gordon and Massive Thirring models.
- (iii) Have understood these solutions in terms of properties of coefficients of (i)