Computers for Mathematics Instruction (CMI) Project

Module 2:

Graphing Software

organised by

written by

Wong Khoon Yoong

**UNIVERSITI
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Introduction

This is the second module of the Computers for Mathematics Instruction (CMI) Project, an in-service program organised by the Ministry of Education, the Secondary Mathematics School-Based Committee, and the Department of Science and Mathematics Education at Universiti Brunei Darussalam. The aim of the CMI Project is to help mathematics teachers acquire the knowledge and develop the skills to use computers to teach mathematics in secondary schools.

This module covers the use of the graphing software, *Graphmatica,* to teach selected mathematics topics in the secondary school mathematics syllabi. Extension examples are also given to challenge the more able pupils.

This module is to be used in a 3-hour training session for teachers. Comments, corrections and any other feedback relating to any part of this manual are most welcome. Please send your comments to me.

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Objectives

At the end of this module, teachers will be able to:

1. use the graphing software together with sample worksheets to teach selected mathematics topics in the classrooms;
2. use the graphing software, together with a screen capture program, as a productivity tool in the preparation of worksheets and tests;
3. understand the pedagogical principles used in the design of the worksheets, so that these principles can be applied to develop own activities later on.

Pre-requisites

Participating teachers should be familiar with Windows 95 and, preferably, have attended Module 1on *Excel* templates.

What is a Graphing Software

A graphing software is a specialised program used by mathematicians in the study of graphs of functions. Most graphing software has the following capabilities:

1. Plot graphs by accepting functions in Cartesian form, $y = f(x)$, $x = g(y)$, as implicit functions (for example, $x^2 + y^2 = 9$), in parametric form, or as polar equations.
2. Allow the user to control the domain and range of values to be plotted.
3. Zoom feature to allow the user to study local and global properties and to obtain more accurate values, especially when solving equations graphically.
4. Trace feature to enable the user to read off the x-y coordinates of points on a given curve.
5. Accept standard mathematical functions.
6. Include calculus features such as plotting of derivative and computing the numerical area under a curve.

There are several commercial software that includes graphing features: *ANU-Graph, Mathematica, MathCad,* and *MathLab*. These commercial packages require large memory and are difficult to learn. In this module, we will use the shareware, *Graphmatica*, by Keith Hertzer. It is powerful and yet very easy to use. It a small program (178 KB in zipped file), so pupils can copy and use it on their home computer, if necessary. As a shareware, it is cheap compared to the above mentioned commercial packages. It can be downloaded from the Math Forum Web site, http://forum.swarthmore.edu. The main limitation of *Graphmatica* is that it cannot plot points entered by the user.

Although this module uses *Graphmatica,* the worksheets given here can be modified to suit any other graphing software (as well as graphing calculators), provided the teachers keep in mind the pedagogical principles involved.

Excel can be used plot graphs, but this requires some programming skills. Thus, a spreadsheet is less user-friendly than a graphing software for plotting purposes.

Installing *Graphmatica*

1. Create a sub-directory in the hard disk and name it *graphma* (if you use a different name, change accordingly in the following steps).
2. Copy the two files, grmat15w.zip and pkunzip.exe, into the sub-directory, graphma.
3. Select Start/Programs/MS-DOS prompt.
4. At the DOS prompt, type: cd \ graphma
5. Note that you are now in the graphma sub-directory. Type:
   pkunzip grmat15w
6. This will unzip the grmat15w.zip file into 16 files. The program file is called graphmat.exe. Close the MS-DOS prompt by clicking on its cross button.
7. To add *Graphmatica* into the taskbar, select Start/Settings/Taskbar. Then select Start Menu Program and click Add. In the Command line box, type: c:\graphma\graphmat.exe or browse to locate and select this file. Click Next and follow the instructions to complete the installation.
8. Once the program has been properly installed, it can be launched from Start/Programs by clicking the *Graphmatica* icon .

9. (Optional) The default look of the *Graphmatica* screen can be changed to suit your preference. I like to change its look as follows. Maximise the *Graphmatica* window. Under View/Colors, select the background to white button, because white background is more legible and uses less ink when printed. To make this change permanent, select File/Save Setup Info.

Overview of Module 2

This manual contain several parts. The first and most important part deals with the use of the graphing software as an instructional tool. Fifteen sample worksheets on various topics are included. The pedagogical principles underlying the design and use of these worksheets are discussed in the Comments sections. Since it is not possible to include here *all* the graph topics, it is hoped that teachers will apply the principles given here to design their own worksheets. The second part briefly explains how to use *Graphmatica*, with or without a screen capture program, as a productivity tool for teachers. The Reference section provides selected readings and some resources on the Internet. The two checklists at the end are to be used to collect data to evaluate the effectiveness of using the technique discussed in this module. Teachers who have conducted lessons based on these worksheets are requested to return the completed checklists to the author.

Instructional Objectives with Graphing Software

The graphing software can plot graphs accurately and rapidly. This power be exploited to achieve several instructional objectives, some of these are also shared by using the *Excel* templates in Module 1.

1. ***To Develop Concepts Through Multi-Modal Linkage.*** Many algebraic concepts are represented in symbols. These symbols can be illustrated *visually* in a graphical mode and *numerically* through substitution. For example, the meaning of the symbol *m* in the linear equation, $y = mx + c$, can be demonstrated by plotting a family of lines. Algebraic and trigonometric identities can be illustrated by superimposing graphs onto one another. Showing this helps to motivate the analytic proof. The graphing software allows the users (teachers as well as pupils) to quickly work with numerous examples and counter-examples to develop the relationships between the symbolic, graphical, and numerical mode of representation. Seeing the connections between these modes is an important aspect of mathematical understanding.

2. ***To Reinforce Concepts.*** In most instances, the teacher will initially explain the concept without the use of computer (see the next section). Once the pupils have acquired some notions of the concept, they can work with the graphing software to explore further examples, without being handicapped by the lack of basic arithmetic skills, tedious calculations, or errors in manual plotting.

3. ***To Rectify Common Errors.*** Pupils often equate expressions that look alike; for example, $(x + 1)^2 = x^2 + 1$ and $\sin 2x = 2\sin x$. By plotting suitable graphs, these pupils can see visually how the expressions are different. The teacher can use this as an additional remedial activity to help pupils rectify their errors and misconceptions. This graphical approach must be reinforced with analytic explanations.

4. ***To Check Graphical and Analytical Solutions.*** Pupils can use a graphing software to check their answers to graphical and non-graphical problems. This develops the desirable habit of self-check in problem solving.

5. ***To Solve Equations Graphically.*** In the real world, there are numerous problems that cannot be solved analytically. In these cases, approximate methods such as graphing are the only possible means to solve the problems. A graphing software is an efficient tool for finding approximate solutions. Pupils should be made aware of this use of graphing technology so that they will appreciate that some mathematical problems cannot be solved by finding the 'right formula' because there is none!

6. ***To Test Conjectures Through Problem Posing.*** Give pupils the opportunity to discover patterns, to explore mathematical properties, and to test conjectures through posing their own "what-if" problems. This is an important process of mathematical thinking. When a graphing software is used for this purpose, pupils are not hampered by their lack of computational or manual plotting ability. However, this discovery mode of learning is best done under the guidance of the teacher.

7. ***To Become Metacognitive.*** Pupils should learn to check their own answers using screen output and to take responsibility for their learning. Group discussions can be incorporated to promote learning by the *constructivist* and *reflective* approach.

8. ***To Acquire Information Technology Skills.*** This is achieved in an indirect way through learning how to use a powerful software.

9. ***To Enhance Motivation to Learn.*** Pupils are generally motivated to learn mathematics using a computer. A graphing software can add interest to mathematics learning because it is easy and fun to use. The graphics outputs can be very vivid and impressive.

Teaching Sequence When Using Graphing Software

The following sections contain 15 worksheets on teaching graphing topics in the secondary mathematics curriculum: Syllabus D, Additional Mathematics and Pure Mathematics. Teacher notes are given to highlight the pedagogical principles underlying each worksheet, so that teachers can apply these principles when they design their own worksheets later on.

To use graphing software effectively, teachers have to provide guidance to the pupils. Unguided, unstructured explorations often take too much time and may not lead to the intended learning objectives, though they may be used occasionally to promote creative thinking. The suggested sequence, as explained also in Module 1, is:

manual plotting ® computer plotting ® manual plotting

A typical teaching sequence is as follows.

1. Explain the intended concept or skill without using computer. This may be done using exposition, a multimodal approach or group discussion. This initial step equips the pupils with the necessary mathematical background to follow the computer-based activity.

2.   Conduct the lesson using the software. This may be done in two ways:

   (a) *Whole Class Lesson.* The teacher demonstrates examples and counter-examples using the software and explains the intended outcomes. This should include question-and-answer to promote active learning. An LCD panel is required to project the computer screen via an overheads projector onto a larger screen so that the whole class can follow the screen display.

   (b) *Pupil Activity.* In this mode, the activity will be conducted in a computer laboratory. Pupils work on the worksheet individually, in pairs, or in small groups. Pair work is the preferred option because discussion among peers will facilitate meaningful learning. Encourage pupils to work with paper, pencil, and calculator to check the screen events. This promotes active learning through "scribbling" and "tinkering". During the lesson, walk around to check that the pupils are on task, and give help whenever necessary. Stop the class periodically to discuss or summarise findings. Collect the worksheets and mark them as class work or homework.

3.   Provide reinforcement. Once the computer activity has been completed, give further practice to reinforce concept and develop the skill to the necessary assessment level.

Caution in Using Graphing Software

Teachers should be aware of the limitations of the computer precision and software algorithms. Given these limitations, some graphs with extreme values (too large or too small) may be wrong or appear distorted. For example, plotting $y = 1000x$ will produce a faint line on the y-axis. The diagram below shows a plot of $y = \cos 30x$, which is quite distorted. The distortion may be rectified by maximising the window or changing the scales.
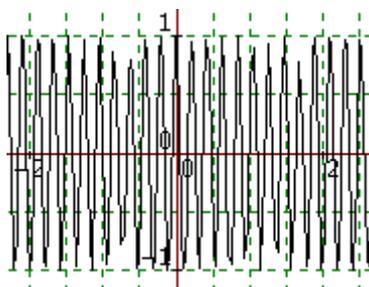
For lower secondary forms, examples must be judiciously chosen to avoid these problems. At upper secondary forms, pupils may explore the effects of these limitations, provided they possess the necessary mathematical knowledge.

Linear Equations

This activity provides an approach to teaching linear equations different from that given in the Lineq *Excel* template in Module 1. Compare the pros and cons of the two approaches. Which approach works better for you and your pupils?

Target Group	Form 2

Content Objectives

1. To relate *m* to the gradient of a line.
2. To relate *c* to the y-intercept of a line.
3. To plot graphs of linear equations given in the slope-intercept form and in the general form.

Process Objectives

1. To explore patterns in families of straight lines. In this approach, we treat each line as a mathematical object.
2. To engage in problem posing and self check.

Comments

1. Work through the worksheet "What is Your Line?" as given in Module 1. This introduces linear equation and the concept of "satisfying an equation" with manual plotting. Manual plotting is used at the initial stage to introduce the required concept *before* pupils use the computer.

2. Work through the worksheet "Lines Parallel to the Axes". *Graphmatica* shows table of values for only integral *x*-coordinates. *Encourage pupils to enter negative values, fractions and decimals in their own examples.*

3. This worksheet also introduces the trace feature of *Graphmatica.* Caution: On some computer screen, it may not be possible to obtain exact values of the points on the line. Experiment with the arrow keys for finer adjustment.

4. Work through the worksheet " *m* is Gradient".

5. *Graphmatica* does not require the asterisks for multiplication. Hence, 2x may be entered as 2x or 2*x.

6. Work through the worksheet "*c* is *y*-intercept".

7. Lead pupils to deduce that as c increases, the lines are parallel and move vertically upwards as in (a). Some pupils may think that the lines move diagonally as shown in (b). Make sure that they do not form this misconception.
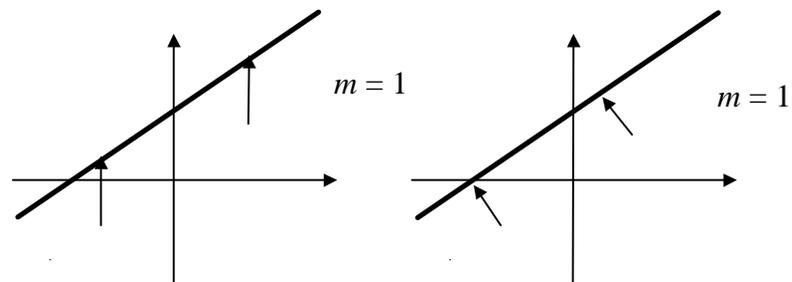
(a) Correct (lines move vertically)   (b) Incorrect (lines move diagonally)

8. Although the syllabus includes only y-intercept, it is appropriate to introduce the term *intercept* for both x- and y-axis. Thus, the x-intercept is the point where the line cuts the x-axis. For the better pupils, provide a project on the intercept-form of the equation of a straight line: $\frac{x}{a} + \frac{y}{b} = 1$. See Item 6 of the worksheet "General Linear Equation".

9. Work through the worksheet "Slope-Intercept Form".

10. Work through the worksheet "General Linear Equation".

11. *Graphmatica* accepts implicitly defined functions. For example, the equation, $2x + 3y + 6 = 0$ may be entered as: $2x + 3y = 6$ or $2x + 3y - 6 = 0$.

Lines Parallel to the Axes

Objective: To plot lines of the form y = constant and x = constant.

1.   Launch the program *Graphmatica* from Start/Programs by clicking the *Graphmatica* icon ![Graphmat].

2.   In the top dialogue box, enter $y = 2$, and press Enter. A line will be plotted. Click Options/Print Tables. This shows the coordinates of some points on the line $y = 2$. Study this table and write a sentence about your observation.

_____

3.   Click the "Coord cursor" button. The cursor becomes a crosshair. Move it along the line and read the coordinates given in the bottom status bar. What do you notice? (On some screen you may not be able to get y = 2 exactly.)

4.   Sketch the following lines and check your answers using *Graphmatica.*

$y = 3$	$y = -4$	$y = 2.4$
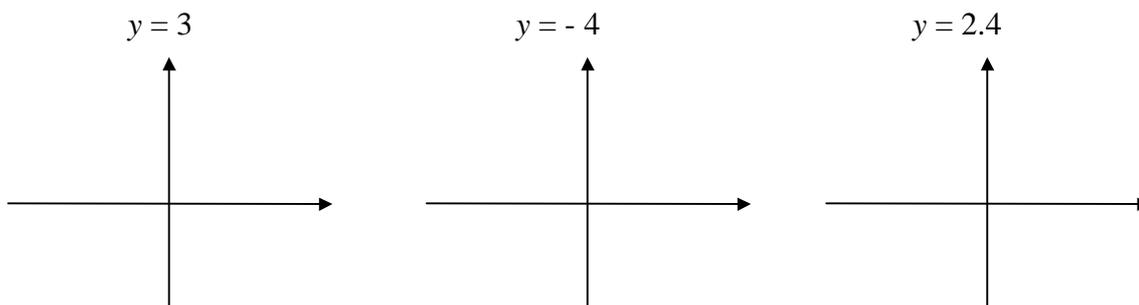
5.   Click the "Clear" button to clear all the graphs. Enter $x = 2$, and press Enter. Study the table of values and use the "Coord cursor" button. What do you notice? Write a sentence about your observation.

_____

6.   Sketch the following lines and check your answers using *Graphmatica.*

$x = 3$	$x = -4$	$x = 2.4$
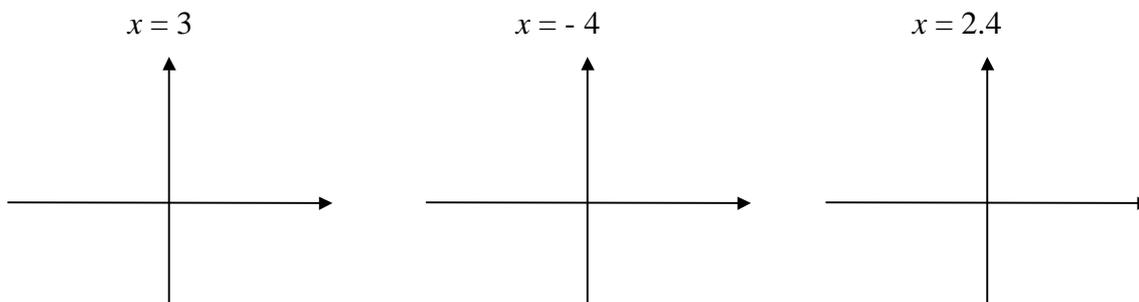
Note: To make the table of values appear and disappear, click Options/Print Tables.

---

m is Gradient

Objective: To relate *m* in the equation $y = mx$ to the gradient of the line.

1.   Begin with a clean screen by clicking the "Clear" button.

2.   Enter y = x, and press Enter. A line will be plotted. Study the table of values and use the "Coord cursor" button to explore points on the line. Write a sentence about your observation.

    _____

3.   Enter $y = 2x$ (or y = 2*x). Study the table of values. Is this line steeper or less steep than the first line?

    _____

4.   Sketch both lines and label them.

5.   Enter $y = 3x$, $y = 4.2x$, and other equations of your own of the form, $y = mx$, where *m* is positive. In each case, sketch the line on the diagram and label it.
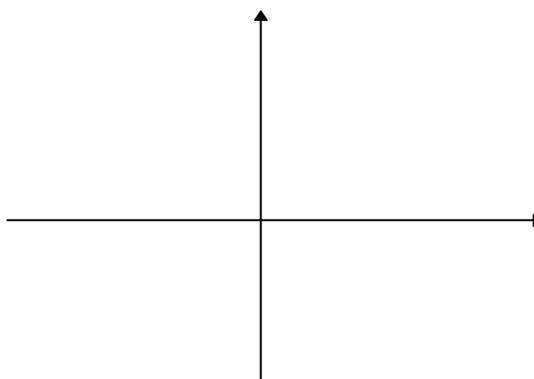
6.   Complete the sentence:

    As *m* increases from 0, the graph _____.

7.   Click the "Clear" button to clear all the graphs.

8.   Enter $y = -0.2x$, $y = -x$, $y = -2x$ and other equations of your own of the form, $y = mx$, where *m* is negative.

    Sketch these lines on the axes given and label them.
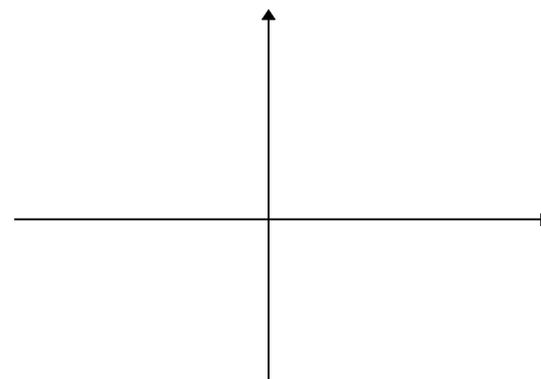
9.   Complete the sentence:

    As *m* decreases from 0, the graph _____.

10.   Complete the following:

        $x =$ a constant, gradient = _____

        $y =$ a constant, gradient = _____

c is *y*-intercept

Objective: To relate *c* in the equation $y = x + c$ to the y-intercept (where the line cuts the y-axis).

1. Begin with a clean screen by clicking the "Clear" button.

2. Enter $y = x$, and press Enter.

3. Enter $y = x + 1$. Study the table of values and use the "Coord cursor" button to explore points on the line. How is this line similar and different from the first line?

   (i)   The two lines are similar: _____

   (ii) The two lines are different: _____

   (iii) The second line can be obtained from the first line by _____

4. Sketch both lines and label them.

5. Enter $y = x + 2$, $y = x + 3$, and other equations of your own of the form, $y = x + c$, where *c* is positive. In each case, sketch the line on the diagram and label it.
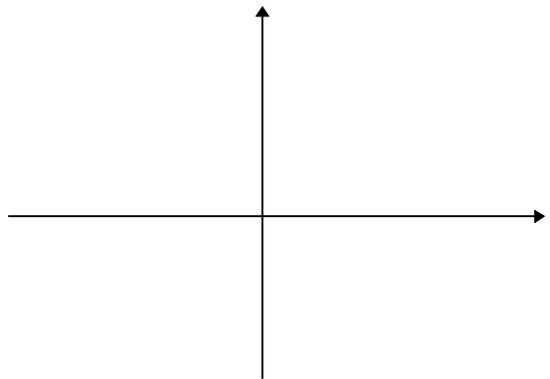
6. Complete the sentence:

   As *c* increases from 0, the graph

   _____.

7. Click the "Clear" button to clear all the graphs.

8. Enter $y = x$, $y = x - 1$, $y = x - 2$, and other equations of your own of the form, $y = x + c$, where *c* is negative.

   Sketch these lines on the axes given and label them.
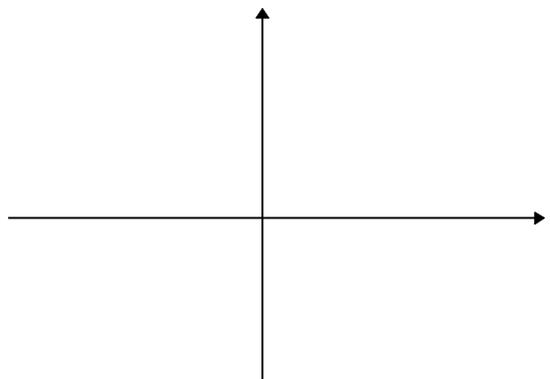
9. Complete the sentence:

   As *c* decreases from 0, the graph

   _____.

Slope-Intercept Form

Objective: To plot lines given by the equation $y = mx + c$.

1.   Without using the computer, sketch the following lines in relation to $y = x$.

   Predicted Sketches

   $y = x - 4$ $y = -2x + 2$ $y = 3x + 1$

   $y = x$ $y = x$ $y = x$
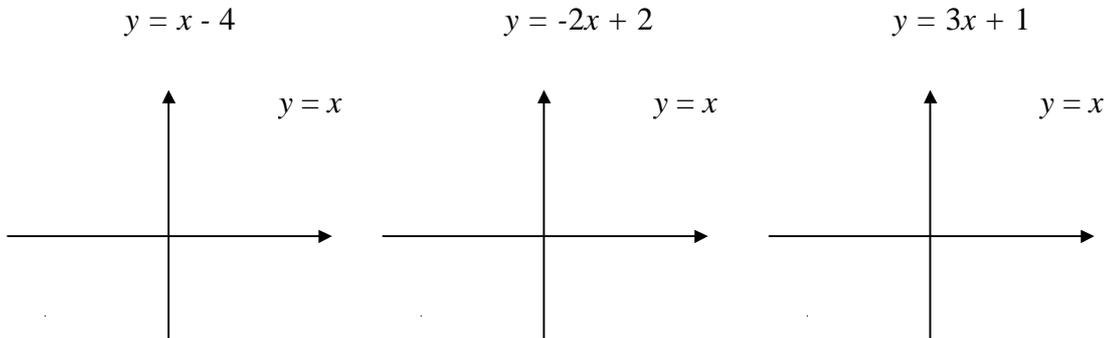
2.   Check your answers using *Graphmatica*.

3.   Repeat the above with other equations of your own. Given several lines on the screen, how do you find out the equation of each line?

   _____

4.   (Optional) If your computer is connected to a printer, select File/Print to print the lines you have plotted, label them and submit it as class work.

5.   Practise with as many examples as you wish until you know how to plot the graph of a given linear equation in the form, $y = mx + c$.

6.   Find the equation of the line that passes through (2, 6) and has a slope of -2. Check your answer using *Graphmatica*.

7.   Write down at least **four** linear equations such that each one will pass through the point (1, 3).

   (a)   (b)

   (c)   (d)

   Enter your equations into *Graphmatica* and check that your lines actually pass through the point (1, 3).

5.   Use linear equations to make some patterns of your own.
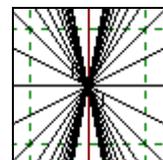
General Linear Equation

Objective: To plot lines given by the general equation $ax + by + c = 0$.

1.   To plot the graph of the general linear equation, $ax + by + c = 0$, make up a table of x-y values for three points. Why choose three points?

   _____

2.   Complete the following table for the equation, $2x + 3y + 6 = 0$.

x	0		1
y		0	

   Why are these three values chosen?

   _____

3.   Plot the above equation on graph paper.

4.   In *Graphmatica,* enter: $2x + 3y + 6 = 0$. Is your graph correct?

   _____

5.   Use *Graphmatica* to check your textbook exercise on plotting.

6.   (a)   Plot the equation $x + 2y = 4$ using *Graphmatica*. Where does the line cut the x-axis and the y-axis?

        It cuts the x-axis at _____ and the y-axis at _____.

        These are called the *intercepts* of the line.

   (b)   Find a line that intersects the above line at the point (4, 0). Check your answer using the computer.

   (c)   Find at least three more lines that intersect line (a) at the point (4, 0). Check your answer using the software. Sketch your lines below. (Optional) Print your answer.
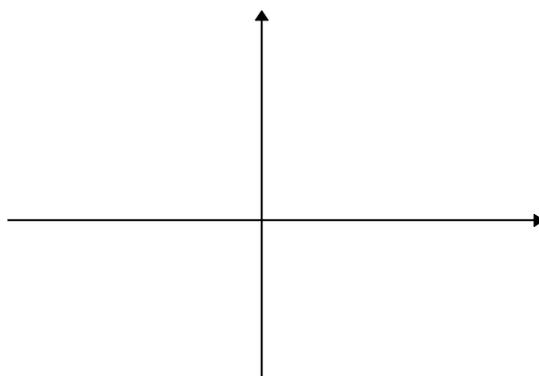
Quadratic Graphs: Line of Symmetry

Target Group   Form 4 Additional Mathematics

Content Objectives
1. To deduce the equation of the line of symmetry of a quadratic graph.
2. To compute the maximum or minimum value of a quadratic function using the turning point.

Process Objectives
1. To spot patterns and test conjecture.
2. To engage in problem posing and self check.

Comments
1. This activity is to be completed after the pupils have some practice of manually plotting simple quadratic graphs.

2. Work through the worksheet "Line of Symmetry".

3. Pupils are expected to

   - deduce the equation of the line of symmetry, $x = -\dfrac{b}{2a}$;
   - note that maximum value is obtained when $a < 0$ and that minimum value is obtained when $a > 0$;
   - calculate the minimum or maximum value by substituting the x-value for the line of symmetry into the quadratic function.

4. Item 3 develops self-check by asking pupils to confirm their answers by plotting the lines of symmetry they have obtained.

5. Item 5 is an example of using graphs to rectify common errors. This remedial activity should be complemented with analytic explanations.

Line of Symmetry

Objective: To obtain the line of symmetry of a quadratic function and to compute its minimum or maximum value.
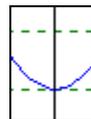
1.   In *Graphmatica* enter the equation, $y = x^2 + 2x$ by typing: x^2+2x.
Where is its line of symmetry? Enter: x = -1. What happens?

_____

2.   Do you agree with the entries in the first row of the table below?

Quadratic Function	a	b	c	Line of Symmetry	Maximum or Minimum Value
$y = x^2 + 2x$	1	2	0	x = -1	min, -1
$y = x^2 + 2x + 6$					
$y = x^2 - 4x$					
$y = x^2 - 4x + 5$					
$y = 2x^2 - 6x - 2$					
$y = 2x^2 + 8x + 2$					
$y = x^2 - 9x + 3$					
$y = 3x^2 + 6x + 5$					

3.   Complete the table above using *Graphmatica* as a tool. Use the "Zoom in" and "Zoom out" buttons to obtain more accurate answers. In each case, plot the line of symmetry to check that your answer is correct.

4.   Answer the following questions.

   (a)   Does the value of c affect the equation of the line of symmetry? Why?

   (b)   What is the rule for finding the equation of the line of symmetry in terms of *a, b,* and *c*? Test your answer with some examples of your own.

   (c)   How can you tell whether a quadratic function has a minimum or a maximum value by looking at the sign of a?

   (d)   How do you use the line of symmetry to *calculate* the minimum or maximum value?

5.   Plot $(x + 2)^2$ and $x^2 + 4$. Is it true that $(x + 2)^2 = x^2 + 4$? Do they have the same line of symmetry and minimum (or maximum) value?

Graphical Solution of Quadratic Equations

Target Group Form 4

Content Objectives

1. To relate the roots of a quadratic equation to the *x*-intercept, i.e., the *x*-coordinates of the points where the curve intersects the *x*-axis.
2. To determine the solutions of two simultaneous equations (quadratic or linear) by finding the *x*-coordinates of the points of intersection.
3. To solve the reverse problem: given the roots of a quadratic equation, find the equation.

Process Objectives

1. To link symbolic mode to visual mode.
2. To engage in problem posing and self check.

Comments

1. This activity is to be completed after the pupils have some practice of solving quadratic equations by manual graphing.

2. Work through the worksheet "Solve Quadratic Equations Graphically".

3. Item 1 uses equations in the factored form only. This is followed by Item 2 which uses equations that cannot be factorized easily. Item 2 illustrates the advantage of a graphical method. Use the Zoom feature to obtain answers correct to the accuracy given. Encourage pupils to zoom in as much as they wish to obtain their "best" solutions.

4. In Item 3, two equations are given to develop the concept of finding the points of intersection of two curves. This is an important general principle which can be applied to solving any equation using a graphical method.

5. Item 4 is the reverse problem with numerous solutions. Let pupils compare their answers. The better pupils may be challenged to give answers involving two equations as in Item 3. Another open-ended question: Find several quadratic curves that pass through the points (-1, 0) and (2, 0). Which of your curves also pass through (0, 3)?

6. When modifying the given examples, include decimals, fractions, equations with identical roots and those without real roots, move the terms around, and so on. This helps to develop flexible thinking.

Solve Quadratic Equations Graphically

Objective: To solve equations graphically by finding the *x*-coordinates of the points of intersection.
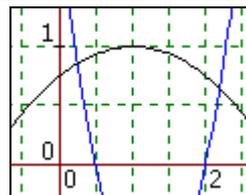
1.   Complete the following table with using *Graphmatica*.

	Quadratic Equation	**Sketch Graph**	**Solutions (Roots)**
(a)	$x^2 - 9 = 0$	$y = x^2 - 9$	$x =$
(b)	$(x - 1)(x + 2) = 0$	$y =$	$x =$
(c)	$3(x + 1.5)(x - 5) = 0$	$y =$	$x =$
(d)	$(3 - x)(2x + 5) = 0$	$y =$	$x =$
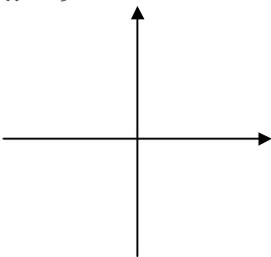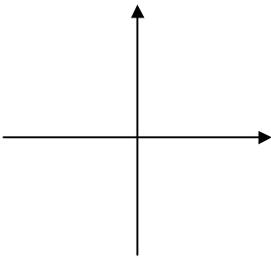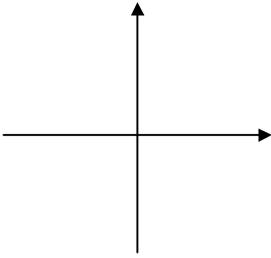
Practise with some examples of your own.

Graphing (Wong)
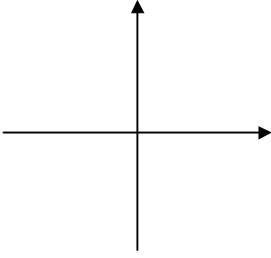
2. The equations below cannot be factorized easily. Use the "Zoom in" button to obtain answers to the accuracy given as well as to your "best" effort.

	Quadratic Equation	**Sketch Graph**	**Solutions (Roots)**
(a)	$x^2 - 3x - 1 = 0$	$y = x^2 - 3x - 1$	Correct to 2 d.p.: $x =$ My "best" solutions: $x =$
(b)	$2x^2 + x - 4 = 0$	$y =$	Correct to 3 d.p.: $x =$ My "best" solutions: $x =$
(c)	$7 - 2x - x^2 = 0$	$y =$	Correct to 2 d.p.: $x =$ My "best" solutions: $x =$
(d)	$x^2 + x + 3 = 0$	$y =$	Correct to 3 d.p.: $x =$ My "best" solutions: $x =$
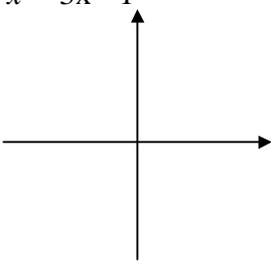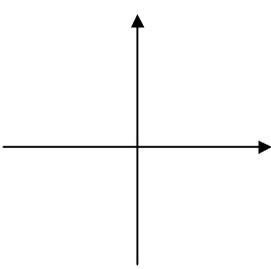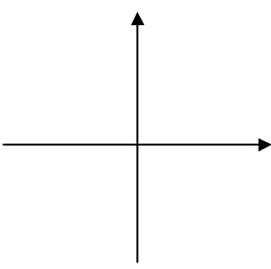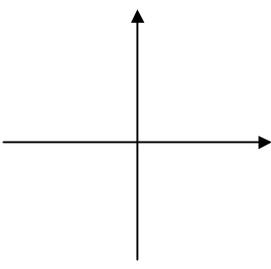
Practise with some examples of your own.

3. Solve the equations below by plotting *two* graphs and finding the *x*-coordinates of the points of intersection. Use the "Zoom in" button to obtain answers as accurately as possible.

	Quadratic Equation	**Sketch Graphs**	**Solutions (Roots)**
(a)	$x^2 - 3x - 1 = x + 2$	$y = x^2 - 3x - 1$ and $y = x + 2$	$x =$
(b)	$x^2 + 2x - 3 = 3x - 1$	$y =$ and $y =$	$x =$
(c)	$4 - x^2 + 2x = 3x^2 - 7$	$y =$ and $y =$	$x =$
(d)	$1 + 2x - x^2 = x^2 - x + 4$	$y =$ and $y =$	$x =$
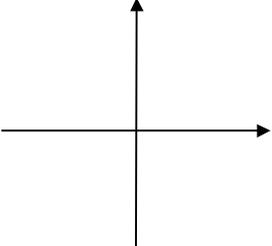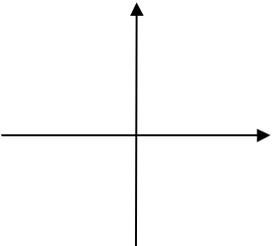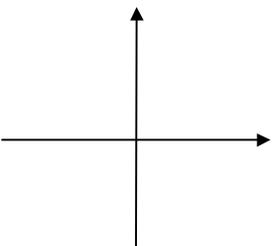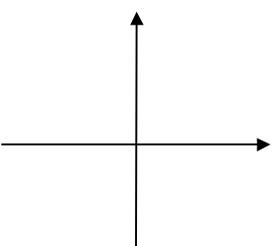
4. Write down as many different quadratic equations as you can whose roots are 2 and 5. Check your answers using *Graphmatica*.

Transformations on Quadratic Graph

Target Group Form 4: Project work for Syllabus D
Form 4: Additional Mathematics

Content Objectives

1. To deduce the effects of three transformations on the quadratic graph, $f(x) = x^2$.
2. To apply these transformations to sketch the graph of $f(x) = a(x + b)^2 + c$.

Process Objectives

1. To explore patterns related to *families* of quadratic curves.
2. To engage in problem posing and self check.

Comments

1. Pupils should be familiar with the function notation.

2. Work through the worksheet "Transforming Quadratic Graph".

3. Pupils are expected to deduce that the graph of

   - $y = f(x) + a = x^2 + a$ is obtained by *translating* the basic graph $y = x^2$ through a units along the y-axis, upward if $a > 0$ and downward if $a < 0$;
   - $y = f(x + a) = (x + a)^2$ is obtained by *translating* the basic graph $y = x^2$ through a units along the x-axis, to the left if $a > 0$ and to the right if $a < 0$;
   - $y = af(x) = ax^2$ is obtained by *stretching* the basic graph $y = x^2$ along the y-axis with factor a; if $a = -1$, the basic graph is reflected in the x-axis.

   The overall effect: $y = a(x + b)^2 + c$
   vertical stretch ⟶
   horizontal translation ⟶   vertical translation
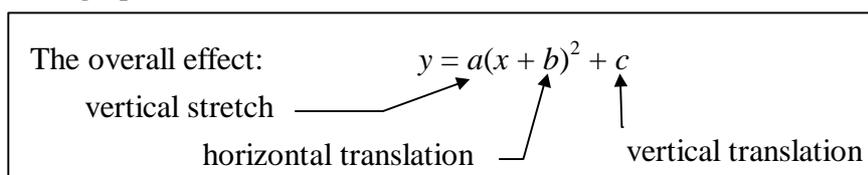
4. The transformation $f(ax)$ is not included here because its effect looks similar to $af(x)$ for the basic quadratic function; it is a *stretch* along the x-axis with factor $\frac{1}{a}$. The matrix for $f(ax)$ is $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$ and for $af(x)$ is $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. Use matrices to show that the effect of $f(\sqrt{a}x)$ is the same as that of $af(x)$ for $f(x) = x^2$.

5. Item 5 uses a short cut to enter a family of equations in *Graphmatica*. Make sure that pupils understand how it works.

6. See also the section on "Transformations on Functions".

Transforming Quadratic Graph

Objective: To apply transformations to sketch quadratic graphs.
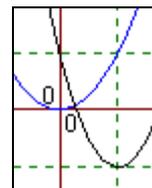
1.   In this activity, the basic quadratic graph refers to $f(x) = x^2$.

2.   Consider $f(x) + a = x^2 + a$. In *Graphmatica,* enter the equations, $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, and other equations of your own of the form, $y = x^2 + a$, where a is positive. What do you notice about the curves? Sketch them and complete the sentence below:

The graph of $y = f(x) + a = x^2 + a$, where $a > 0$, can be obtained from the graph of $y = f(x) = x^2$ by

_____
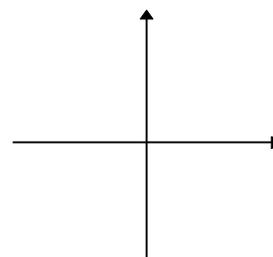
3.   Clear the screen and enter the equations, $y = x^2$, $y = x^2 - 1$, $y = x^2 - 2$, and other equations of your own of the form, $y = x^2 + a$, where a is negative. What do you notice about the curves? Sketch them and complete the sentence below:

The graph of $y = f(x) + a = x^2 + a$, where $a < 0$, can be obtained from the graph of $y = f(x) = x^2$ by

_____
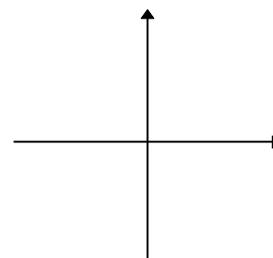
4.   In Items 2 and 3 above, a is called the *parameter*. Every value assigned to the parameter gives one equation and one curve only. When you vary the parameter, you obtain a *family* of curves.

5.   (Optional) Short cut: *Graphmatica* has a simple way of entering a *family* of equations using parameter. Clear the screen and enter: $y = x^2+a$ {a: 0, 6, 1}. What do you notice? Try $y = x^2+a$ {a: 0, 6, .2} and $y = x^2+a$ {a: -6, 0, 1}. What happens in each case?

6. Consider $f(x + a) = (x + a)^2$. By entering suitable equations, complete the sentences below:

    (a) The graph of $y = f(x + a) = (x + a)^2$, where $a > 0$, can be obtained from the graph of $y = f(x) = x^2$ by

    _____

    (b) The graph of $y = f(x + a) = (x + a)^2$, where $a < 0$, can be obtained from the graph of $y = f(x) = x^2$ by

    _____

7. Consider $-f(x) = -x^2$. By entering suitable equations, complete the sentence below:

    The graph of $y = -f(x) = -x^2$ can be obtained from the graph of $y = f(x) = x^2$ by

    _____

8. Consider $af(x) = ax^2$. By entering suitable equations, complete the sentences below:

    (a) The graph of $y = af(x) = ax^2$, where $a > 0$, can be obtained from the graph of $y = f(x) = x^2$ by

    _____

    (b) The graph of $y = af(x) = ax^2$, where $a < 0$, can be obtained from the graph of $y = f(x) = x^2$ by

    _____

9. Draw the graph of $y = x^2 - 2x$ using *Graphmatica*. Sketch it in each diagram below using blue colour. For each diagram, sketch the new curve given using red colour; show clearly how each curve is related to the graph of $y = x^2 - 2x$. Check your answers using *Graphmatica*.

    $y = x^2 - 2x + 1$ $y = 3x^2 - 6x$ $y = (x + 2)^2 - 2(x + 2)$
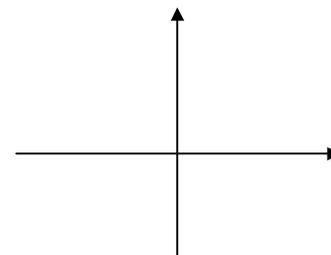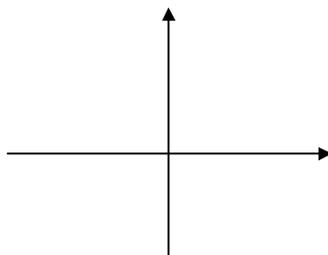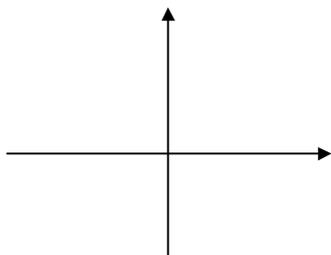
10. Use the above ideas to sketch the graphs of the following:

    (a) $y = (x - 2)^2 + 3$ (b) $y = 2(x + 1)^2 + 4$ (c) $y = (3 - x)^2 - 1$

Linear Inequalities in Two Variables

This activity introduces the shading of a region defined by linear inequalities in two variables. *Graphmatica* shades the region defined by an inequality. It does NOT distinguish between a *strict* inequality ($<$ or $>$) and an inequality ($\leq$ or $\geq$). It accepts inequalities given in different forms, such as $y > 2x + 1$, $2x + y > 1$, $2x + y - 1 < 0$, and so on.

Target Group   Form 5

Content Objectives
1. To shade the region defined by linear inequalities.

Process Objectives
1. To engage in problem posing and self check.

Comments

1. Work through the worksheet "Solve Linear Inequalities".

2. Item 1 begins with the simple case, $x < 3$. At lower form, the solution is the set of numbers less than 3. This is represent on the number line below:
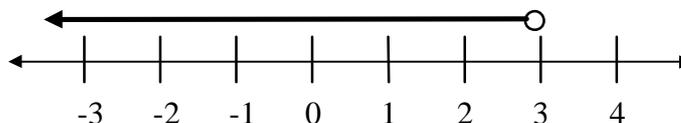
In the present case, the solution refers to the coordinate plane involving two variables (x and y). It is important to explain this difference to the pupils to avoid possible confusion. The context of the problem will make it clear which answer is expected.

3. Item 2 uses the *trace* feature of *Graphmatica* to stress the meaning of the inequality.

4. For Item 9, pupils may notice that $y < mx + c$ is defined by the region "below" (or to the left of) the line, whereas $y > mx + c$ is given by the region "above" (or to the right of) the line. Such observation is useful, but pupils should not memorise it as a mechanical rule. The best strategy for determining which region is required is by testing with specific points *not* on the given equation. It is useful to begin with the origin, if possible.

5. Item 10 illustrates regions defined by parallel lines.

6. Item 12 has many different answers. Let pupils compare their answers.

Solve Linear Inequalities

Objective: To solve linear inequalities involving two variables by shading.
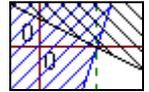
1.   Enter the inequality $x < 3$. Copy the screen output in the grid below. *Draw a dotted line for the boundary defined by $x = 3$. Why?*
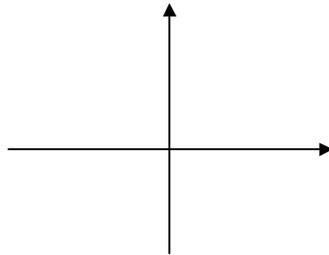
2.   Click the "Coord cursor" button. The cursor becomes a crosshair. Move it inside the shaded region and notice the coordinates printed at the left-hand bottom corner. All the points inside the shaded region have x-coordinates less than 3. What happens to the coordinates when you move the crosshair

     (a) inside the *unshaded* region?   _____

     (b) on the line $x = 3$?   _____

3.   Shade the region defined by the following inequalities. Check your answers using *Graphmatica.* Draw *dotted lines* for the boundaries. Why?

     $x > 2$ $x < -4$ $x > 2.4$
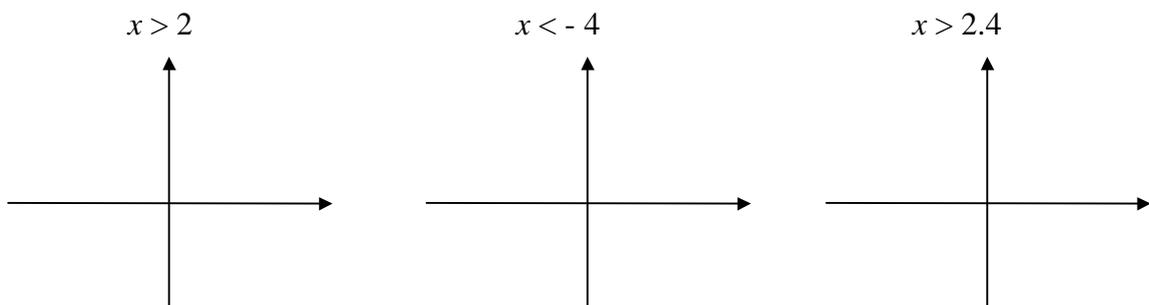
4.   *Graphmatica* cannot plot inequalities involving ≤ or ≥. If these inequalities are given, you have to draw a *solid* line for the boundary. Why? Complete the following without using *Graphmatica*.

     $x \geq 2$ $x \leq -4$ $x \geq 2.4$
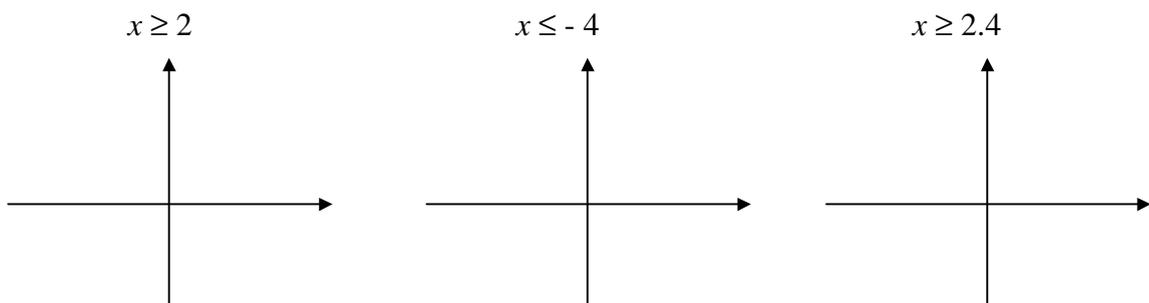
5. Shade the region defined by the following inequalities. Check your answers using *Graphmatica.*

$y > 2$ $\qquad\qquad\qquad$ $y < -4$ $\qquad\qquad\qquad$ $y \geq 2.4$
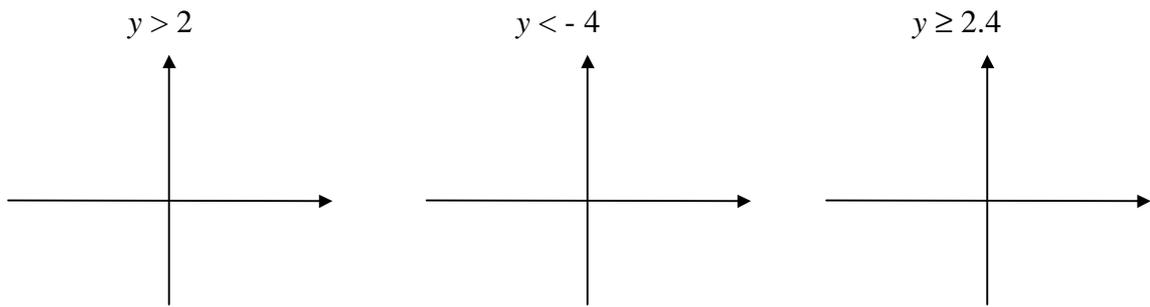
6. Enter $x > 2$ and $y < 3$. Copy the screen output in the grid below. The region which is shaded twice is called the solution of the two inequalities. Use the "Coord cursor" button to explore the coordinates of the points inside this double shaded region.
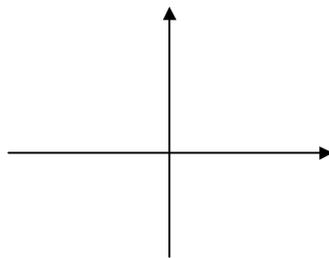
7. Enter $y < 2x$. Copy the screen output in the grid below. Use the "Coord cursor" button to explore the coordinates of the points inside this region.
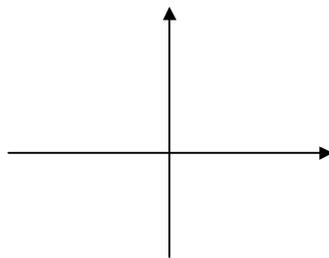
8. Shade the region defined by the following inequalities. Check your answers using *Graphmatica.*

$y > 2x + 1$ $\qquad\qquad$ $y < 3x - 6$ $\qquad\qquad$ $y > 5 - x$
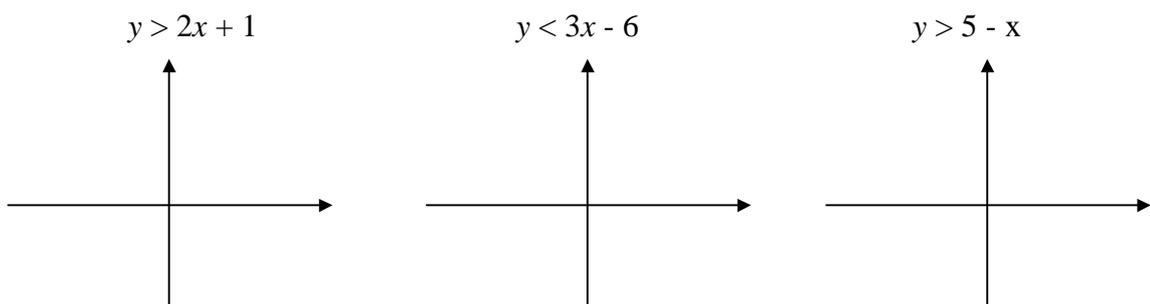
9. Given an inequality such as $y < mx + c$ or $y > mx + c$, can you find a way to decide which region is the solution?

10. Enter $2x + y < -6$. Copy the screen output in the grid below.
Next enter $2x + y < -4$, $2x + y < 0$, $2x + y < 3$, $2x + y < 6$, and so on. What do you notice about these inequalities?
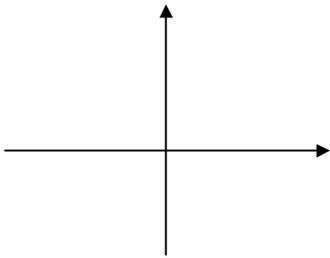
Observation:

11. Determine the regions defined by the inequalities given. Check your answers using *Graphmatica.*

(a) $x + 2y > 1$ and $3x - y < 3$

(b) $y - x > 3$, $2x + 3y - 6 < 0$ and $y > -2$

12. Find **three** linear inequalities such that the region defined by the three inequalities contains the point (1, 3).

(a)

(b)

(c)

Check your answer using *Graphmatica*.

Inverse Functions

Target Group   Form 4 Syllabus D and Additional Mathematics

Content   1.   To plot graphs of inverse functions by equation and by
Objectives   reflection in the line $y = x$.

Process   1.   To spot pattern and test conjecture.
Objectives   2.   To engage in problem posing and self check.

Comments   1.   Pupils should know that: given a function $y = f(x)$, its inverse
                        can be obtained by interchanging x and y in the function and
                        writing y in terms of x. Graphically, this means that the point
                        (x, y) becomes (y, x). This concept is reviewed in Items 1 to 4.

                   2.   Work through the worksheet "Graphs of Inverse Functions".

                   3.   Item 5 leads to the reflection property: the graph of f^{-1} is
                        obtained by reflecting the graph of f in the line $y = x$.

                   4.   To plot the inverse functions of Item 7, interchange x and y
                        and enter directly into *Graphmatica.* For example, (a) involves
                        plotting: y=2/x+1 and x=2/y+1; it is NOT necessary to find the
                        inverse function in the form $y = f(x)$.

Graphs of Inverse Functions
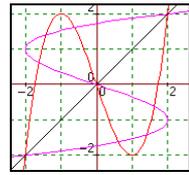
Objective: To plot graph of inverse functions.

1.   Complete the following table for the function, $y = 2x - 1$.

x	-3	0	1	2	3
y					

2.   Use an algebraic method to determine the inverse of the function, $y = 2x - 1$.

3.   Use your answer to (2) to complete the following table for the inverse function.

x					
y	-3	0	1	2	3

4.   Notice that the inverse function can be obtained by interchanging x and y in the given function. Explain why this is so.

_____

5.   Plot the graphs of the function, its inverse, and the line $y = x$ using *Graphmatica*. Sketch these graphs below. What do you notice about these graphs?
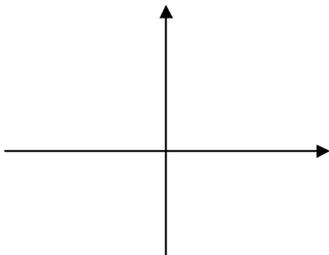
<div align="center">Observation:</div>

6.   Check your observation using other functions of your own.

7.   For each function below, sketch its graph and that of its inverse. Check your answers using *Graphmatica.* Plot also the line $y = x$.
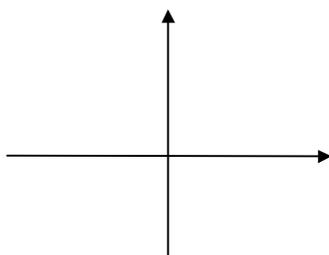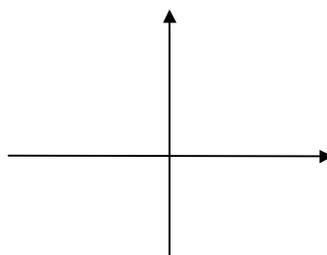
(a) $y = \frac{2}{x} + 1$ (b) $y = x^2 + 3$ (c) $y = \frac{2x-1}{x+1}$

Exponential Functions

Target Group Form 4 Syllabus D

Content Objectives

1. To plot graphs of exponential functions $y = a^x$ where a is a positive integer.
2. To solve exponential equations graphically.

Process Objectives

1. To make conjectures about properties of exponential functions and test conjectures.
2. To engage in problem posing and self check.

Comments

1. Revise Form 2 work on indices. For example, compute the values of $27^{\frac{2}{3}}$, - $27^{\frac{2}{3}}$, $(-27)^{\frac{2}{3}}$ without using calculator and the value of $30^{\frac{2}{3}}$ using the x^y key on a calculator.

2. Work through the worksheet "Graphs of a^x".

3. For Item 6, pupils should note:

    *   for $x < 0$, the value of y decreases as the base, a, increases;
    *   for $x > 0$, the value of y increases as the base, a, increases;
    *   these curves do not intersect the x-axis;
    *   for each curve, the y-values increase very rapidly.

4. For Item 7, *Graphmatica* interprets -2^x as the same as (-2)^x, which is different from -(2^x). This is noted in Item 9. The function $(-2)^x$ is not defined when x involves even root. For example, $(-2)^{\frac{1}{2}} = \sqrt{-2}$ is undefined (has no real value), although $(-2)^{\frac{1}{3}} \approx$ -1.26. Thus, an exponential function with negative base is not defined. For Items 7 and 9, it is crucial to use brackets correctly.

5. Item 10 uses the technique of superimposing graphs to check equality. (a) is false; (b) is true by Law of Indices. In (b), the graphs of $y = 2^x \times 3^x$ and $y = 6^x$ coincide; select Options/Print Tables to check that the values are also identical for both functions, $2^x \times 3^x$ and 6^x.

Graphs of a^x

Objective: To plot graph of exponential function, $y = a^x$, where a is a positive integer.

1. Complete the following table for the exponential function, $y = 2^x$.

x	-3	-2	-1	0	1	2	3
y	0.125			1			

2. Plot the above equation on graph paper.

3. In *Graphmatica,* enter: y = 2^x. Compare the graph on the screen with your graph. Do you get the same answer?

    _____

4. Click Options/Print Tables to compare the computer values with those in your table. What do you notice?

    _____

5. Click the "Coord cursor" and move the crosshair over the curve and note the coordinates of the point given at the left-hand bottom corner. Is it possible to obtain negative *y*-value on this curve?

    _____

6. Use *Graphmatica* to plot the graphs: $y = 3^x$, $y = 4^x$, $y = 5^x$, and so on. Copy the screen output in the grid below and describe your observation about these graphs.
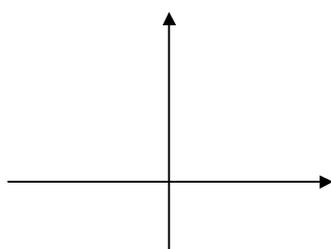
    Observation:

7. Use *Graphmatica* to plot the graph of $y = -2^x$ by entering: y = -(2^x). Be careful to type the brackets correctly as shown. How is this graph related to that of $y = 2^x$?

    _____

8.   Repeat Question 6 with the graphs: $y = -3^x$, $y = -4^x$, $y = -5^x$, and so on. Copy the screen output in the grid below and describe your observation about these graphs.
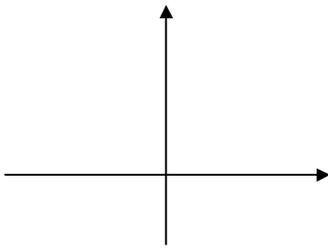
Observation:

9.   (a)   Enter: y = -2^x. *Graphmatica* treats -2^x as (-2)^x = $(-2)^x$. What happens? Can you explain the result?

_____

     (b)   Is $(-2)^x = -2^x$?

_____

10. (a)   Is it true that $2^x + 3^x = 5^x$? Check your conjecture by plotting suitable graphs using *Graphmatica.*

     (b)   Is it true that $2^x \times 3^x = 6^x$?

11. Use *Graphmatica* to solve the following equations as accurately as possible:

     (a) $3^x = 15$

     (b) $2^x = 3x + 2$

     (c) $3^x + 4^x = x^2$

The Exponential Function

The exponential function refers to $y = e^x$, where the base e is called the Euler's constant.

Target Group	Form 5 Additional Mathematics

Content Objectives
1. To illustrate the meaning of $e \approx 2.71828$ graphically.
2. To plot graph of the exponential function.

Process Objectives
1. To link symbolic mode to visual mode.
2. To engage in problem posing and self check.

Comments

1. Revise the work on exponential functions in Syllabus D.

2. Work through the worksheet "Meaning and Graph of e^x".

3. For Item 2, introduce the term *asymptote* with reference to the line $y \approx 2.718$.

4. For Item 3, note that the printed values become constant at 2.7181. This is due to the accuracy of the software. Pupils need to learn to be cautious about computer display. Their ability to interpret such display will improve with their mathematical knowledge.

5. Make designs without the axes!
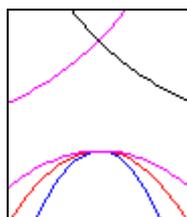
Objective: To illustrate graphically the meaning of the Euler's constant and to plot graph of the exponential function, $y = e^x$.

1. The Euler's Constant, e, is defined as the limit of the expression $\left(1 + \frac{1}{x}\right)^x$ as x becomes infinitely large (approaches infinity, or $x \to \infty$). Use calculator to complete the following table of values for $y = \left(1 + \frac{1}{x}\right)^x$.

x	1	2	10	100	1000	10 000	1 000 000
y	2						

2. Use *Graphmatica* to plot the graph of $y = \left(1 + \frac{1}{x}\right)^x$. Copy the screen output in the grid below and describe your observation about this graph.
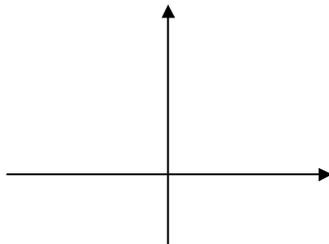
Observation:

3. Click "Zoom out" a few times and note the changes to the graph. Next click Options/Settings/Change range and change the Left value to 0, the Right value to 10 000, the Top value to 3, and the Bottom value to 2. Click Options/Print Tables. Explain what you notice about the graph and the printed values.

_____

4. Clear the screen and plot the graph of the exponential function by entering: y = exp(x) or e^x. Sketch the graph below.
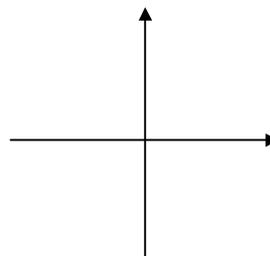
5. Sketch the following graphs. Use *Graphmatica* to check your answers.

(a) $y = -e^x$ (b) $y = e^{-x}$ (c) $y = 3e^{-x}$ (d) $y = e^x + e^{-x}$

Sine Functions

This activity covers the graphs of sine functions in degrees and in radians. By default, *Graphmatica* calculates trigonometric functions in radians; e.g., sin (30) ≈ -0.988 where 30 refers to 30 radians (≈ 1718^0). To refer to degrees, include *d with the variable; for example, sin (30*d) = 0.5. Note that d is *Graphmatica*'s built-in constant that converts degrees to radians.

Target Group   Form 4 Additional Mathematics

Content Objectives	1.	To plot graphs of sine functions in degrees.
	2.	To determine the amplitude, period and phase of a sine function.
Process Objectives	1.	To link symbolic mode to visual mode.
	2.	To engage in problem posing and self check.
Comments	1.	Pupils should be able to construct table of values of sin (x), where $-360^0 \leq x \leq 360^0$ and plot its graph. Note the periodicity of the function and its maximum and minimum values.
	2.	Work through the worksheet "Graphs of Sine Functions in Degrees".
	3.	For Item 2, stress that the maximum (or minimum) value occurs at infinitely many values of x.
	4.	In Item 3, the effect of a (amplitude) in $y = a\sin x$ is to stretch the graph of $y = \sin x$ along the y-axis. The y-value at a given point is obtained by multiplying the y-value for sin x by a.
	5.	In Item 4, the effect of p (period) in $y = \sin px$ is to stretch the graph of $y = \sin x$ along the x-axis, thus changing its *period* to $\frac{360}{p}$ (in degrees).
	6.	In Item 5, the effect of q (phase) in $y = \sin (x + q)$ is to translate the graph of $y = \sin x$ along the x-axis, q units to the left if $q > 0$ and to the right if $q < 0$. Note that sin $(x + 90^0)$ is the same as cos x. Show this by plotting y=cos(x*d); the two graphs will coincide.
	7.	Once the pupils have understood the above ideas, enter: $y = a*sin(x*d)$ {a: -5,5,1} to plot a family of sine curves quickly. Repeat for $y = \sin((a*x)*d)$ {a: -5,5,1} and $y = \sin((x+a)*d)$ {a: -5,5,1}.
	8.	Repeat the above activity for cos x and tan x.

Graphs of Sine Functions in Degrees

Objective: To plot graphs of sine functions in degrees.

1.   Click View/Grid Range to change the Left value to -360, the Right value to 360, the Top value to 3, and the Bottom value to -3. Then enter: y=sin(x*d). Compare the computer display with your graph of $y = \sin x$. Click Options/Print Tables to study some values.

2.   Click "Zoom out". How many maximum values and minimum values are shown, and where do these values occur? Use "Zoom in" to obtain more accurate *x*-values. Complete the table below.

	Maximum	Minimum
Value =		
Occurs at x =		
Other possible values of x		

Repeat the "Zoom out" several times and note what happens.

3.   Re-set the grid range to -360 to 360. Enter: y=sin(x*d), y=2sin(x*d), y=3sin(x*d), and so on. Copy the screen output in the grid below and describe your observation about these graphs.
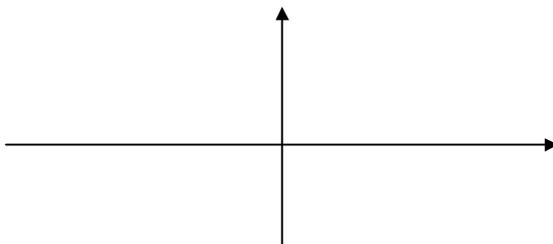
Observation:

4.   Repeat the above by entering: y=sin(x*d), y=sin(2x*d), y=sin(3x*d), and so on.

Observation:

5.   Repeat the above by entering: y=sin(x*d), y=sin((x+30)*d), y=sin((x+45)*d), y=sin((x+90)*d) and so on.

Target Group	Form 4 Additional Mathematics

Content Objectives	1.	To plot graphs of sine functions in radians.
	2.	To determine the amplitude, period and phase of a sine function.

Process Objectives	1.	To link symbolic mode to visual mode.
	2.	To engage in problem posing and self check.

Comments	1.	Pupils should be able to construct table of values of $\sin(x)$, where $-2\pi \le x \le 2\pi$ and plot its graph.
	2.	Work through the worksheet "Graphs of Sine Functions in Radians".
	3.	For Item 2, stress that the maximum (or minimum) value occurs at infinitely many values of x.
	4.	In Item 3, the effect of a (amplitude) in $y = a\sin x$ is to stretch the graph of $y = \sin x$ along the y-axis. The y-value at a given point is obtained by multiplying the y-value for $\sin x$ by a.
	5.	In Item 4, the effect of p (period) in $y = \sin px$ is to stretch the graph of $y = \sin x$ along the x-axis, thus changing its *period* to $\frac{2p}{p}$ (in radians).
	6.	In Item 5, the effect of q (phase) in $y = \sin(x + q)$ is to translate the graph of $y = \sin x$ along the x-axis, q units to the left if $q > 0$ and to the right if $q < 0$. Note that $\sin(x + \frac{p}{2})$ is the same as $\cos x$. Show this by plotting y=cos(x); the two graphs will coincide. To enter π in *Graphmatica*, type pi.
	7.	Once the pupils have understood the above ideas, enter: y = a*sin(x) {a: -5,5,1} to plot a family of sine curves quickly. This can be repeated for y = sin(a*x) {a: -5,5,1} and y = sin((x+a) {a: -5,5,1}.
	8.	Item 6 gives an example of solving complex equation graphically where there is no analytic solution.
	9.	Item 7 introduces *Graphmatica* Trig paper, which marks the x-axis in multiples of π. However, for Additional Mathematics, it is better to work with the Rectangular graph paper because this is used in the examination questions.
	10.	Repeat the above activity for $\cos x$ and $\tan x$.

Graphs of Sine Functions in Radians

Objective: To plot graphs of sine functions in radians.

1. If you have changed the grid range, click the "Default grid" button. Then enter: y=sin(x) or y=sin x (leave a space in front of x). Compare the computer display with your graph of $y = sin\ x$. Click Options/Print Tables to study some values.

2. Click "Zoom out". How many maximum values and minimum values are shown, and where do these values occur? Use "Zoom in" to obtain more accurate x-values. Complete the table below.

	Maximum	Minimum
Value =		
Occurs at x =		
Other possible values of x		

Repeat the "Zoom out" several times and note what happens.

3. Enter: y=sin(x), y=2sin(x), y=3sin(x), and so on. Copy the screen output in the grid below and describe your observation about these graphs.
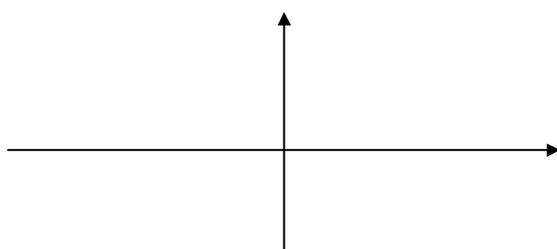
Observation:

4. Repeat the above by entering: y=sin(x), y=sin(2x), y=sin(3x), and so on.

5. Repeat the above by entering: y=sin(x), y=sin(x+π/6), y=sin(x+π/4), y=sin(x+π) and so on.

6. Solve graphically: $x\ sin\ x = 0.5$.

7. (Optional) Click View/Graph Paper and choose Trig. What happens?

Trigonometric Identities

This activity illustrates trigonometric identities visually. Visualisation enhances intuitive understanding and should be introduced before the algebraic proofs.

Target Group	Form 4 Additional Mathematics

Content Objectives	1.	To relate trigonometric identities to graphs.
Process Objectives	1.	To spot patterns.
	2.	To resolve misconceptions through visualisation.
Comments	1.	Explain that an identity is always true for the given domain.
	2.	Work through the worksheet "Trigonometric Identities".
	3.	Items 2 and 3 lead to the Pythagoras identity.
	4.	Item 4 explores other identities using a similar approach..
	5.	For Items 5 and 6, the technique of superimposing graphs is used to help pupils resolve their misconceptions about trigonometric identities. Having worked through this activity, pupils must still learn the algebraic proofs.

Trigonometric Identities

Objective: To understand trigonometric identities visually.
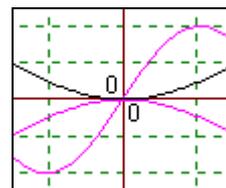
1. In this activity, angles are given in radians. Set Options/Print Tables on so you can study the values together with the graphs.

2. Enter: y=(sin x)^2; this plots $y = \sin^2 x$. Enter: y=(cos x)^2; this plots $y = \cos^2 x$. Sketch both curves below.
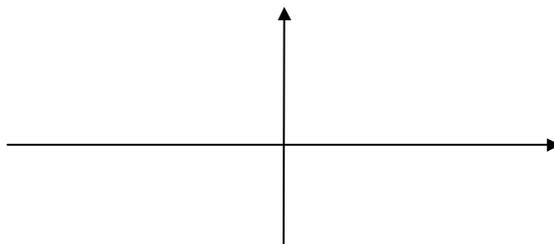
3. $y = \sin^2 x + \cos^2 x$ can be obtained by "adding" the two curves together. Sketch it in the grid above. Check your answer by entering: y=(sin x)^2 + (cos x)^2. What is the name of this identity?

_____

4. Use the above approach to illustrate the following identities.

    (a) $1 + \tan^2 x = \sec^2 x$ (b) $1 + \cot^2 x = \operatorname{cosec}^2 x$
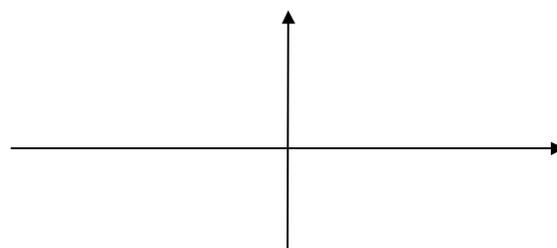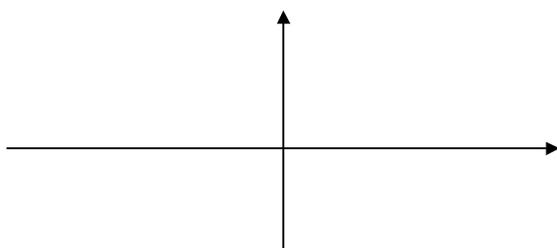
5. (a) Is it true that $\sin 2x = 2\sin x$?
       Plot $y = \sin 2x$ and $y = 2\sin x$. What can you deduce from the graphs?

    (b) Next plot $y = 2\sin x \cos x$ on the same grid.
        Write down your conclusion.

        _____
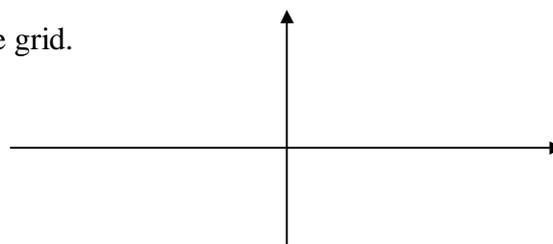
6.   (a)   Is it true that $\cos 2x = 2\cos x$?
             Plot $y = \cos 2x$ and $y = 2\cos x$. What can you deduce from the graphs?

      (b)   Next plot $y = 2\cos^2 x - 1$ on the same grid.
             Write down your conclusion.

             _____
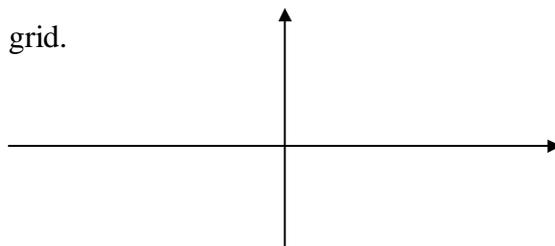
7.   Use *Graphmatica* to check whether the following is true or false. Explain the answer using trigonometry.

	Identity?	Sketch Graph	True or False?
(a)	$\sin (x - \pi) = \cos x$		
(b)	$\cos (x + 2\pi) = -\cos x$		
(c)	$\cot x + \tan 2x =$ $\cot x \sec 2x$		
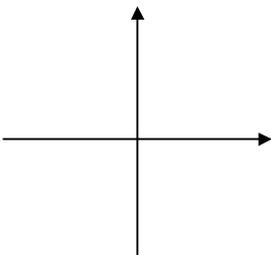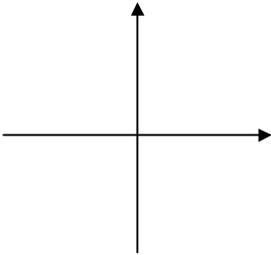
Absolute Valued Functions

Target Group   Form 4 Additional Mathematics

Content
Objectives
1.   To plot graphs of absolute valued functions.
2.   To determine the effect of taking the absolute value of a given function.

Process
Objectives
1.   To link symbolic mode to visual mode.
2.   To engage in problem posing and self check.

Comments
1.   Work through the worksheet "Graphs of Absolute Valued Functions".

2.   Item 1 introduces the definition of the absolute valued function, $|x|$ and its plotting from first principles.

3.   From Items 4 and 5, pupils should deduce that the vertex of the graph $y = |mx + c|$ has coordinates $\left(-\frac{c}{m}, 0\right)$. However, it is not necessary to remember this rule.

4.   For Item 6, pupils should spot the pattern: taking the absolute value of a function will reflect the negative y-values about the x-axis.

5.   The absolute valued function is the first example of a piecewise function that pupils encounter in secondary schools. In *Graphmatica,* such piecewise functions can be plotted by entering the rules for the separate domains: y=-x {,0} and y=x {0,}. The domain {,0} includes all real values less than 0, and the domain {0,} includes all values ≥ 0. Another example:

$$f(x) = \begin{cases} x + 6, \text{ for } x < -2 \\ x^2, \text{ for } -2 \leq x < 2 \\ 8 - 2x, \text{ for } x \geq 2 \end{cases}$$

Enter as three separate functions: y=x+6 {,-2}, y=x^2 {-2,2}, y=8-2x {2,}.

6.   Challenge. Plot graphs $|x| + |y| = 4$ and $x^2 + y^2 = 16$. Calculate the area enclosed between the two curves. Note that *Graphmatica* accepts implicitly defined equations; for example, the first graph can be entered as: abs(x)+abs(y)=4.

Graphs of Absolute Valued Functions

Objective: To plot graphs of absolute valued functions.
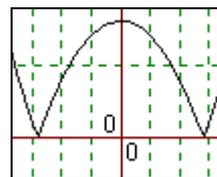
1.   The absolute valued function is defined as follows:

$$|x| = \begin{cases} -x, \text{ for } x < 0 \\ x, \text{ for } x \geq 0 \end{cases}$$

Complete the following table of values for $|x|$ and plot it on graph paper.

x	-4	-3	-2	-1	0	1	2	3	4
$|x|$									

Describe the shape of this graph.

_____

2.   Enter the absolute valued function as: y=abs(x). Do you get the same graph?

_____

3.   Enter: y=abs(x), y=2abs(x), y=3abs(x), and so on. Sketch the graphs below and describe their shapes.
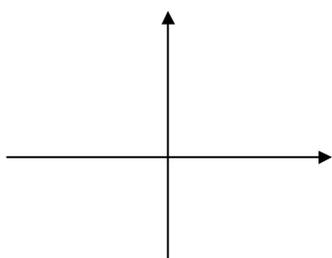
Observation:

4.   Enter: y=abs(x), y=abs(x+1), y=abs(x+2), and so on. Sketch the graphs below and describe their shapes.
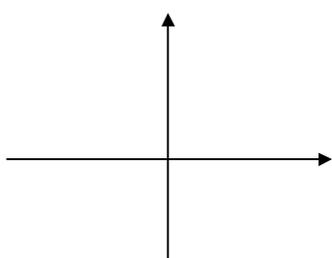
Observation:

5. Complete the table below and determine the vertex of the graph $y = |mx + c|$.

	Function	Sketch Graph	Vertex
(a)	$y = |2x + 1|$ Enter: y=abs(2x+1)		
(b)	$y = |3x - 4|$		
(c)	$y = |5 - 2x|$		
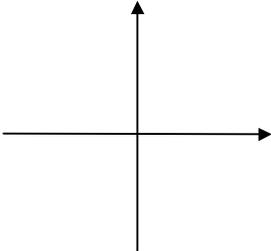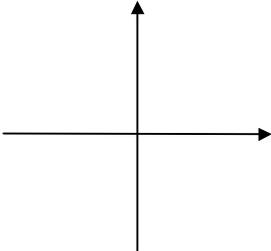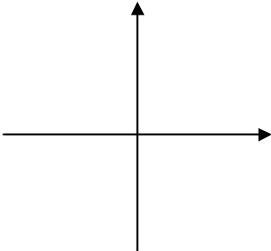
Write down a rule for finding the coordinates of the vertex of the graph $y = |mx + c|$ and test your rule with further examples of your own. Explain your rule algebraically.

Rule: _____

Algebraic Proof:

6. Work out the following examples and use *Graphmatica* to check your answers. What is the effect of taking the absolute value of a function?

	Function	My Answer	*Graphmatica*
(a)	$y = x + 1 - 3$ versus $y = |x + 1| - 3$		
(b)	$y = 2x + 1$ versus $y = |2x + 1|$		
(c)	$y = x^2 - 4$ versus $y = |x^2 - 4|$		
(d)	$y = x^3$ versus $y = |x^3|$		
(e)	$y = \sin x$ versus $y = |\sin x|$		
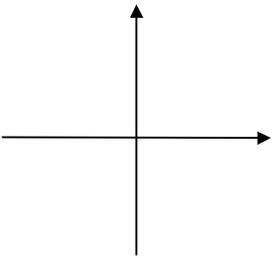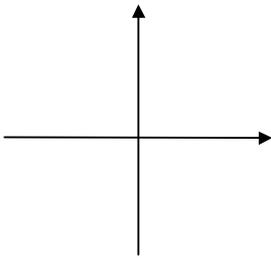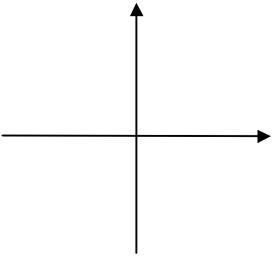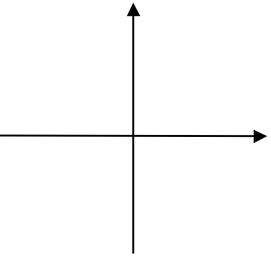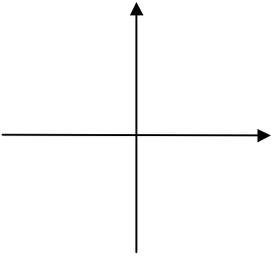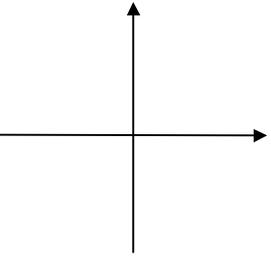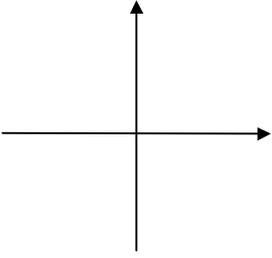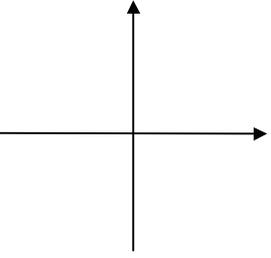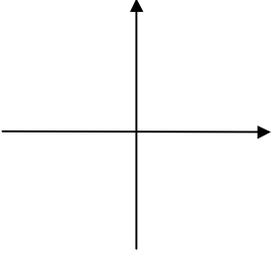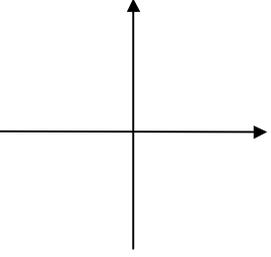
Conclusion:

Transformations on Functions

This activity follows from the earlier activity "Transformations on Quadratic Graphs".

Target Group   Form 4: Additional Mathematics

Content Objectives

1.   To deduce the effects of various transformations on given function.

Process Objectives

1.   To o explore patterns related to *families* of curves.
2.   To engage in problem posing and self check.

Comments

1.   Pupils should be familiar with the function notation.

2.   Work through the worksheet "Transforming Functions".

3.   The effects are:

    (a) $f(x) + a$: *translate* the basic graph through a units along the y-axis, upward if $a > 0$ and downward if $a < 0$;

    (b) $f(x + a)$: *translate* the basic graph through a units along the x-axis, to the left if $a > 0$ and to the right if $a < 0$;

    (c) $af(x)$: *stretch* the basic graph along the y-axis with factor a; if $a = -1$, the basic graph is reflected in the x-axis;

    (d) $f(ax)$: *stretch* the basic graph along the x-axis with factor $\frac{1}{a}$; if $a = -1$, the basic graph is reflected in the y-axis;

    (e) $|f(x)|$: the negative y portion of the basic graph is reflected in the x-axis;

    (f) $f(|x|)$: no simple description!

    (g) $f^{-1}(x)$: *reflect* the basic graph along the $y = x$. In many cases, the formula for $f^{-1}(x)$ cannot be written as simple formula. However, its graph can be plotted by interchanging x and y. See the section "Inverse Functions".

3.   Repeat the above using other functions (simple ones only).

4.   Item 3 is a test whether pupils can apply the above transformations even though the original function is not known. The answer is (-3, 5), irrespective of the original function $f(x)$.

Transforming Functions

Objective: To explore the effects of various transformations on given function.
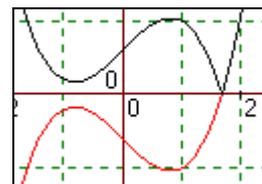
1. Let f(x) = x^3 - 3x. Plot its graph.

In the following table, *a* is the parameter that can take various values: positive, negative, zero, integers, fractions, etc. Experiment with different values of *a* in order to understand the effects of each transformation. Use *Graphmatica* to plot the graphs for the values of *a* you wish to explore, so that you can focus on what the transformation is doing. Complete the table below.

	Transformation	**Graph**	**Effect**
(a)	f(x) + a = x^3 - 3x + a		
(b)	f(x + a) =		
(c)	a f(x) =		
(d)	f(ax) =		
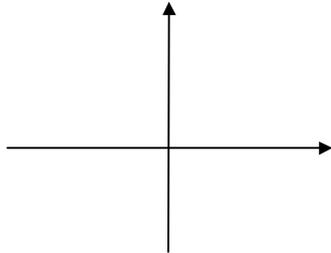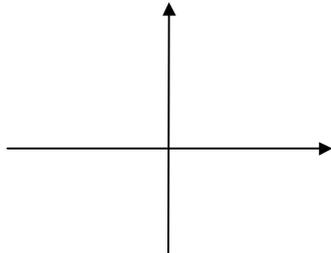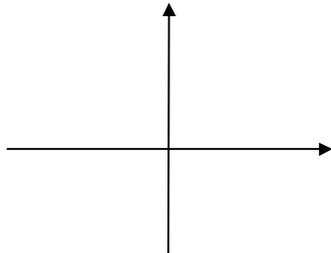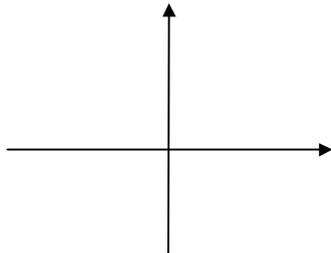
	Transformation	Graph	Effect
(e)	$|f(x)| =$		
(f)	$f(|x|) =$		
(g)	$f^{-1}(x) =$		
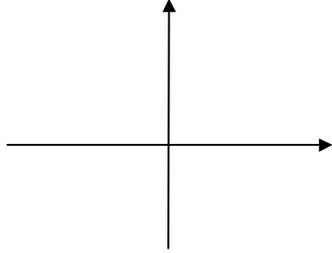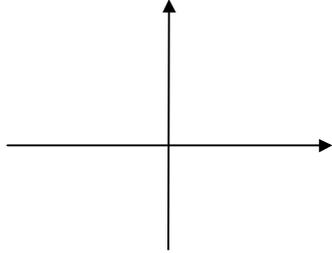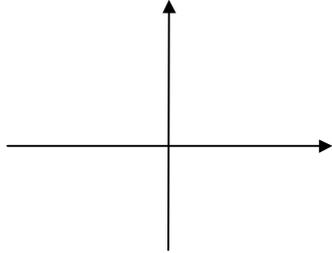
2. Repeat the above with other functions, for example, f*(x)* = sin *x*.

3. The point (-2, 4) is on the graph of f(*x*). What are its coordinates under the graph of g(*x*) = 2f(*x* + 1) - 3 ? You may like to explore with several graphs f(*x*) that pass through the point (-2,4). Does the answer depend on which function you assign to f(*x*)?

Derivatives of Functions

Target Group Form 5 Additional Mathematics

Content Objectives
1. To understand the graphical relationship between a function and its first and second derivatives.
2. To compare and contrast between curves.

Process Objectives
1. To link symbolic mode to visual mode.
2. To engage in problem posing and self check.

Comments
1. Pupils should know that the (first) derivative gives the equation of the tangent to the function.

2. Work through the worksheet "Derivatives of Functions".

3. Item 2 helps pupils to focus on the key features between the curves. See the suggested answers below. If pupils deduce something different from the table, discuss their answers in class.

4. Item 4 (f) gives an error message; $|x|$ cannot be differentiated at the origin because of the sharp turn there. If one keeps zooming in at any point on a smooth curve like $y = x^2$, the curve eventually becomes straight. However, if this is done for the vertex of $|x|$, no matter how many times one zooms in, the sharp turn is always there. Try this on *Graphmatica*! This is the difference between points that have a derivative against those without a derivative.

5. For further practice, begin with a function, say x^4, and keep finding its successive derivatives. What happens?

	Function: $y = x^2$	First Derivative: $\frac{dy}{dx} = 2x$	Second Derivative: $\frac{d^2y}{dx^2} = 2$
Degree	2	1	0 (constant)
Shape	curve	straight line	horizontal line
Turning point	minimum at $x = 0$	equals 0	positive
Decreasing		negative	
Increasing		positive	

Derivatives of Functions

Objective: To understand the graphical relationship between a function and its first and second derivatives.
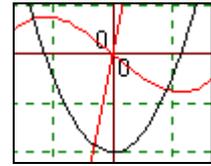
1. Let $y = x^2$. Then $\frac{dy}{dx} = 2x$. Enter: y=x^2. Click Calculus/Find Derivative. Notice that the derivative is given at the bottom status bar and its graph is plotted. Copy the graph and its derivative below using different colours.
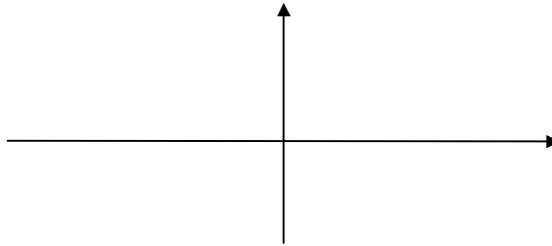
2. Complete columns 2 and 3 of the table below to compare the shapes of the two graphs.

	Function: $y = x^2$	First Derivative: $\frac{dy}{dx} = 2x$	Second Derivative: $\frac{d^2 y}{dx^2} =$
Degree	2		
Shape	curve		
Turning point			
Decreasing			
Increasing			

3. Click on the derivative. Click Calculus/Find Derivative. Look at the status bar and note the new graph. This is the second derivative of the function. Add this graph to the grid above using different colour. Complete column 3 of the table above.

y=2 ' deriv. of y=2*x ' deriv. of y=x^2

4. Repeat the above for the functions below.

    (a) $y = x^2 - 6x + 5$ (b) $y = -x^2 - 6x + 5$

    (c) $y = x^3 - 3x$ (d) $y = \sin x$

    (e) $y = \frac{x}{x^2 + 1}$ (f) $y = |x|$

Parametric Equations

Target Group   Form 5 Additional Mathematics

Content Objectives	1.	To plot graphs of simple parametric equations.

Process Objectives	1.	To link symbolic mode to visual mode.
	2.	To engage in problem posing and self check.

Comments	1.	Pupils should have some ideas about parametric equations. Graphs of parametric equations are not normally plotted, though adding a visual aspect will enhance understanding.
	2.	Work through the worksheet "Graphs of Parametric Equations".
	3.	Item 1 reinforces the concept of *parameter* through manual plotting, that is, each value of the parameter, t, corresponds to one point on the curve.
	4.	Item 2 explains how to enter parametric equations in *Graphmatica.* The domain must be specified; the x and y equations must be separated by a semicolon; the parameter must be in t and not other variable.
	5.	Item 3 provides further practice in plotting parametric equations to stress the meaning of *parameter.*
	6.	Item 4 is an investigation that requires the use of problem solving strategies.
	7.	Item 5 leads to the standard result that the inverse of a graph is obtained by reflecting it about the line $y = x$.
	8.	For extension, obtain the parametric equations of standard curves.

Graphs of Parametric Equations

Objective: To plot graphs of parametric equations.

1.   A curve is defined by the following parametric equations: $x = 3t, y = t + 2$.

Complete the following table of values and plot the points on graph paper.

t	-3	-2	-1	0	1	2	3
x							
y							

   (a)   What curve do you get?   _____

   (b)   What is its Cartesian equation?

2.   Enter the above equations in *Graphmatica:* x=3t ; y=t+2 {-3,3}. Click Options/Print Tables. Do you get the same values and graph?

   _____

3.   For each set of parametric equations below, draw a table of values and graph it. Use *Graphmatica* to check your answer.

   (a) $x = 2t, y = t^2 + 1, -3 \le t \le 3$ (b) $x = 3t, y = \frac{3}{t}, -4 \le t \le 4$

   (c) $x = t(1 + t), y = t^2(2 + t), -3 \le t \le 3$ (tricky!)

   (d) $x = 2\sin t, y = 3\cos t, 0 \le t \le 4\pi$ (Enter pi for π in *Graphmatica*)

4.   (Investigation) Consider the parametric equations: $x = 1 + at, y = 2 + bt$, where $0 \le t \le 4$, and *a, b* take values -1, 0 or 1.

   (a)   Construct a table to cover all the nine possible cases.
   (b)   In each case, plot the graph.
   (c)   How is the slope of each line segment related to the values of *a* and *b*?
   (d)   Repeat with $-4 \le t \le 0$.

5.   Inverse functions can be plotted using parametric equations.

   (a)   Plot $y = x^3 + x$ by the parametric equations: $x = t, y = t^3 + t, -4 \le t \le 4$.
   (b)   Its inverse is obtained by replacing *x* by *y* and vice versa, i.e., $x = y^3 + y$. How would you enter this as parametric equations? Try it.
   (c)   Plot $y = x$. What do you notice?

Power Series

Target Group	Form 6 Principal Mathematics	
Content Objectives	1.	To illustrate graphically the behaviours of power series of standard functions.
Process Objectives	1.	To link symbolic mode to visual mode.
Comments	1.	Several standard functions have well-known power series obtained by Taylor's formula. The behaviours of these power series can be studied by plotting the expansion terms by terms. This will also highlight the domain where the series converges.

2. Consider the power series for $\sin x$:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots, \text{ valid for all } x.$$

The plots around (-2, 2) are shown below. As more terms are included, the series approaches the sine curve.
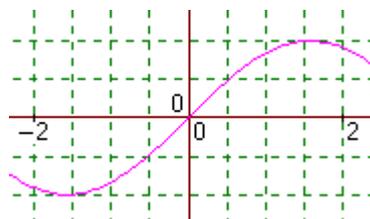
(a) $\sin x$ (b) $\sin x$ and x
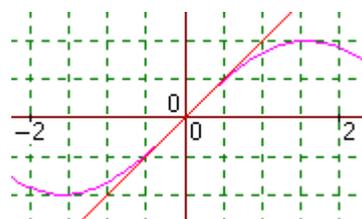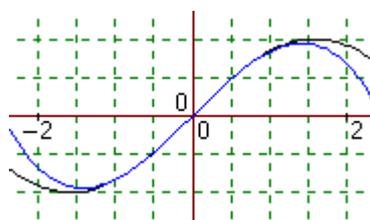
(c) $\sin x$ and $x - \frac{x^3}{6}$ (d) $\sin x$ and $x - \frac{x^3}{6} + \frac{x^5}{120}$
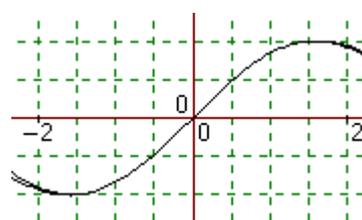
3. Apply the above method to other power series.

Graphing Software as a Productivity Tool

Graphmatica can be used as a productivity tool in the following ways to save time and effort and to enhance the quality of printed materials.

1. The graphs can be copied and pasted into worksheets and tests. This can be done in three different ways.

    (a) To copy the whole screen, select Edit/Copy Graphs BMP/Monochrome. This will copy the screen to the clipboard as a bitmap of size about 40 KB. Open the file containing the worksheet (or a new file) in Winword, and click Edit/Paste. If the coloured version is used, the size increases to about 670 KB!

    (b) To copy the graph on a smaller window, adjust the size of the *Graphmatica* window by dragging from the bottom right hand corner; the cursor becomes a double-sided arrow. Continue as in (a).

    (c) To copy a small area of the window, use a screen capture utility. A useful one is the shareware called *Scap* by Allty Systems. The Capture/Area feature is most useful. To install this utility, copy the scap.exe file into a sub-directory and then click on it.

    Another screen capture utility is the shareware *All Screen 95 PRO* (screen95.exe), which is available from: http://web.ukonline.co.uk/a.mccann. It has a 30 days trial period.

2. Use the graphing software to check answers to exercises. This saves time and effort for the teachers.

3. Prepare special files that contain graphs and comments so that pupils can load and study them. Follow the following steps.

    (a) Prepare the graphs.
    (b) Click Show Labels.
    (c) Click Labels/Title to add overall title to the graphs.
    (d) Click Labels/Annotate to add labels or comments to the graphs. After entering the text, click Place to position the text.
    (e) Click File/Save and enter a name for the file. *Graphmatica* saves the file as an equation list with the extension .GR.

    To view an equation list, click File/Open List and select the file. Try the sample lists included in the program and the three examples, *wongabs.gr*, *wonginv.gr*, and *sine.gr*.

References

Other than *Graphmatica*, the following two graphing software are worth exploring.

The *GraphEq* from Pedagoguery Software is a commercial software. A demo version can be downloaded from: http://www.peda.com/grafeq/. It can show animation of graphs. It costs US $100 to purchase.

The *Winplot* program (*winplotz.exe*) is available at: http://www.math.msu.edu/Help/ftp or http://www.exeter.edu/~rparris. This shareware can plot 3-D graphs. Click on the file to install it.
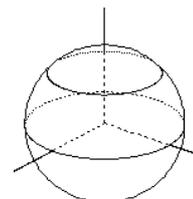
Selected references of research and teaching ideas about graphing software and graphics calculators are given below.

Bloomfield, A. & Harries, T. (Eds.). (1995). *Teaching, learning and mathematics - with I.T.* Derby: Association of Teachers of Mathematics. (QA20.C65 T423. The book contains many good examples of how computer can be used in mathematics teaching.)

Dugdale, S. (1990). Beyond the evident content goals Part III: An undercurrent-enhanced approach to trigonometric identities. *Journal of Mathematical Behavior, 9*(3), 233-287.

Dugdale, S., Wagner, L.J., & Kibbey, D. (1992). Visualizing polynomial functions: New insights from an old method in a new medium. *Journal of Computers in Mathematics and Science Teaching, 11*, 123-141.

Froelich, G. (Ed.). (1994). *Technology in the classroom* (Vols. 1-2). Lexington, Mass: COMAP. (QA20.C65 T423. Interesting examples covering various aspects of teaching mathematics with computers.)

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior, 7*, 135-173.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics, 21*(5), 431-450.

Tall, D. (1985). Understanding the calculus. *Mathematics Teaching, 100*, 49-53.

Wong, K.Y. (1990). A graphing program using Logo. *Journal of Computers in Mathematics and Science Teaching, 9*(2), 85-91.

Wong, K.Y. (1997). Learning graphs and geometric transformations with computer: An experience with primary pre-service students. In M. Quigley & P.K. Veloo (Eds.), *Innovations in science and mathematics curricula* (pp. 326-334). Bandar Seri Begawan: Universiti Brunei Darussalam.

The following journals are useful resources for locating teaching ideas and research findings about graphing in mathematics instruction.

- *Mathematics in Schools* (S QA11.A1 M37)
- *Mathematics Teachers* (S QA1 M37)
- *Mathematics Teaching* (S QA11.A1 M353)
- *Micromath* (S QA76.95 M52)
- *School Science and Mathematics* (S Q1 S28)
- *Teaching Children Mathematics* (S QA135.5 T42)

Teacher Feedback

Name: _____

School: _____   Class: _____

Date of Lesson: _____

For each item, circle the code that best indicates your degree of agreement or disagreement to the item.

Strongly Disagree **(SD)**
Disagree **(D)**
Uncertain **(U)**
Agree **(A)**
Strongly Agree **(SA)**

1.	The graph worksheet was easy to use.	**SD**	**D**	**U**	**A**	**SA**
2.	I enjoyed using the graphing software to teach the topic.	**SD**	**D**	**U**	**A**	**SA**
3.	I liked to use the graphing software to prepare worksheets and tests.	**SD**	**D**	**U**	**A**	**SA**
4.	I was not sure of some of the features on how to use the graphing software.	**SD**	**D**	**U**	**A**	**SA**
5.	I would like to try another lesson using graphing software.	**SD**	**D**	**U**	**A**	**SA**
6.	I could cover the same material more effectively without using this graphing software.	**SD**	**D**	**U**	**A**	**SA**
7.	After this lesson, I felt more confident in using computer to teach mathematics.	**SD**	**D**	**U**	**A**	**SA**
8.	Most of my pupils could complete the activity as given.	**SD**	**D**	**U**	**A**	**SA**

Describe briefly any difficulties you faced in using this graphing software and the worksheet in your lesson.

Pupil Feedback

Name: _____   Sex: _____

Class: _____   Date of Lesson: _____

In Computer Studies Class?   Yes   No

Have Computer at Home?   Yes   No

For each item, circle the code that best indicates your degree of agreement or disagreement to the item.

Strongly Disagree **(SD)**
Disagree **(D)**
Uncertain **(U)**
Agree **(A)**
Strongly Agree **(SA)**

1.	The worksheet was easy to use.	**SD**	**D**	**U**	**A**	**SA**
2.	I enjoyed using computer to learn this topic.	**SD**	**D**	**U**	**A**	**SA**
3.	The software was easy to use.	**SD**	**D**	**U**	**A**	**SA**
4.	I was confused by the instructions on how to use the software.	**SD**	**D**	**U**	**A**	**SA**
5.	I would like my teacher to use this method to teach the next topic.	**SD**	**D**	**U**	**A**	**SA**
6.	Doing the activity was a waste of time.	**SD**	**D**	**U**	**A**	**SA**
7.	I prefer my teacher to tell me exactly what to do rather than to do the activity.	**SD**	**D**	**U**	**A**	**SA**
8.	I liked to discuss things with my friends.	**SD**	**D**	**U**	**A**	**SA**
9.	The computer activity helped me to understand better.	**SD**	**D**	**U**	**A**	**SA**
10.	I found the computer activity challenging.	**SD**	**D**	**U**	**A**	**SA**

Write down all the things that you did not like about this lesson.